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I. ECONOMIC BACKGROUND

1. Time Value of Money; discounting

Recall first the definition of *simple interest*, at $100x\%$ p.a. With one unit (a pound, say) invested for one year, at the end of the year we have:
$1 + x$ with interest calculated yearly;
$(1 + \frac{x}{2})^2$ with interest calculated twice-yearly;
$(1 + \frac{x}{n})^n$ with interest calculated (or *compounded*) n times per year (12 for monthly, 365 for daily, etc.).

We would like the compounding to be done as often as possible (so as to be able to exploit the essence of compound interest – getting interest on the interest). To do this, we use

$$(1 + \frac{x}{n})^n \uparrow e^x \qquad (n \to \infty) \qquad (x > 0):$$

exponential growth is the limit of compound interest as the interest is compounded continuously.

Note. 1. Exponential growth (or decay) occurs widely in nature, as well as in economics/finance! – e.g.

Growth of biological populations. A species expanding into a new environment with no limit on resources can and does grow exponentially (rabbits in Australia, bacteria overwhelming a weakened organism, etc.).

Radioactive decay. The amount of material decays exponentially (the rate is measured by the *half-life* – the time it takes for half of what is left to decay).

2. Exponential growth is extremely rapid, and hence is *destabilizing.* This is one of the underlying causes behind financial crises (such as the Credit Crunch, or Crash of 2008). The wonder is that the financial world is as stable as it is, most of the time.

3. Possibly for this reason, there were religious prohibitions on lending money at interest – 'usury', as the (Christian, Catholic) Church called it in the Middle Ages (there are still some such reservations in parts of the Islamic world). However,

4. Without the inducement of interest, it is difficult to justify taking the risk of lending money. Without lending to finance investment and business activity, credit dries up, business activity slows down and the economy shrinks (as recently).

Discounting.

With economic activity driven by money lent and borrowed at interest, one must distinguish between prices in nominal terms and prices in real terms, whenever any length of time has elapsed. Otherwise, one is not comparing like with like. (This gives a good way to distinguish 'good' from 'bad' politicians, with an election pending!) We do this by *discounting*. With r the current short-term interest rate (short rate), one discounts over a time-interval of length t by a factor e^{-rt}. In much of the mathematics below, we will *discount everything*, and if S_t denotes a price at the present time t, its real value at a later time T in present (time t) prices is $e^{-r(T-t)}S_t$. This is done for accounting purposes in Net Present Value (NPV) calculations. These are also used as a tool in assessing whether or not an investment should be made. We will not pursue this in detail here: part of the message of Real Options (VI.6) is that the NPV approach to investment decisions is misleading.

For most purposes, we use as interest rate the rate for *riskless* lending/borrowing. Of course, this overlooks a lot of complications! For example:
1. *Borrowing v. lending.* Traditionally, banks made their money on the difference between the interest they paid to depositors on their savings accounts, and the higher rate they charged borrowers, whether private customers on overdrafts or businesses on borrowing for investment. However, banks not only lend, they borrow, from each other (LIBOR = London Inter-Bank Offer Rate, etc)[1] The difference between such borrowing and lending rates introduces *friction* into the markets. One often (e.g., in the standard Black-Scholes theory below) neglects this, and works with an idealized market.
2. *Risky v. riskless.* Major government debt (Treasury bonds in the US, gilts in the UK) is traditionally regarded as riskless. However, governments do occasionally default on their debts (Mexico and Russia, within fairly recent memory). Banks also occasionally default (though government institutions – the Fed in US, the Bank of England in UK – may intervene as 'lender of last resort' to rescue a troubled institution – e.g. Northern Rock, UK, 2008).
3. *Fixed rate or variable.* Interest rates vary, and one may need to reflect this by using a function $r(t,T)$ rather than a constant r.
4. *Stochastic or deterministic.* It may be that the variability $r(t,T)$ is better

[1] In the recent LIBOR scandal – known inevitably as the Lie-bor scandal – it emerged that individual traders in the participating banks (which only included the large and reputable ones) were systematically distorting the quotes at which they purported to lend or borrow. This is a blatant form of market manipulation; it has led to large fines already, and to heavy (and expensive) reputational damage to the banks.

modelled as random, or *stochastic* – see Ch. III below, and VI.7.6.
5. *Term structure of interest rates*. The mathematics of this – the *term structure of interest rates* – is very interesting and important. But it is (a lot) harder than most of the mathematics we shall be doing here.
6. *Stock markets v. money markets (bond markets)*. There are two principal kinds of financial markets, stock markets and money markets. In stock markets, what are traded – bought and sold – are stocks and shares – part-ownership in a company, both its physical assets, its intangible assets (goodwill, brand name etc.) and its future earnings, which may be released to shareholders as *dividends*. Investors may buy for capital appreciation, dividends or both. The prototype of the relevant mathematics is the *Black-Scholes theory*, of which more below in Ch. IV. On the other hand, in money markets, money is being borrowed (usually by governments, or large companies) over time, to be repaid later at specified rates of interest and/or with specified payments or *coupons*.

The borrower binds himself to repay with interest, and the agreement to do so is called a bond, hence *bond markets* (Shylock, *Merchant of Venice*: I will have my bond!). Bonds issued by leading governments are regarded as safe, or 'gilt-edged', hence the term *gilts* in the UK. Their US equivalents are called Treasury bonds, or T-bonds.

2. Economics and Finance; Utility

Economics is largely concerned with questions such as supply and demand (for everything: commodities, manufactures, capital, labour, etc.). Much of economics deals with how prices are determined.

Finance is concerned with the borrowing and lending of money needed to engage in economic activity. We will be dealing here with the more mathematical side of this. In (mathematical) finance, one takes prices largely as given, and deals with questions such as how to price options (real or otherwise). Thus Finance is a part of Economics – and a fairly small and specialized part at that.

Aside. The Industrial Revolution began in the UK, c. 1760-1850. Under the influence of this, the Agricultural Revolution, the pre-eminence of the Royal Navy and the rise of the British Empire, the UK became the dominant world power in the late 1800s/early 1900s. This was dissipated during the two World Wars, but even c. 1950-80 the UK economy was heavily engaged in large-scale manufacturing industry: steel, textiles, coal, ship-building, volume cars, locomotives, etc. In the years 1980-2010, the old 'smoke-stack' industries largely disappeared, leaving an economy largely based on the service industries on the one hand, and on the financial sector and the housing market on the other. This rather unbalanced economy has left the UK vulnerable in the aftermath of the recession of 2008 on.

Small economic agents are *price takers*. They have no power to influence prices, which they can either take or leave – but equally, do have the power to enter the market without thereby moving the market against them. By contrast, large economic agents are *price makers*. They do have the power to influence prices – but against this, are visible, and so are vulnerable, when forced to enter the market through weakness.

Trading.

The price of common everyday items is accurately known at any given time. Anyone trying to sell at a higher price than the 'going rate' would tend to lose market share to cheaper competitors, and eventually have to reduce towards the going rate or go out of business. At the other extreme, items never bought and sold do not have a price – are literally priceless (Buckingham Palace, Westminster Abbey, the Houses of Parliament, ...).

In between the two, prices are known but not accurately – to within some interval. This is the *bid-ask spread* – the gap between the price at which a market participant will buy, and the (higher) price at which he will sell.

When large trades are made, prices jump. This is because the large trade affects the current balance between supply and demand, and the price is the level at which markets clear – that is, at which supply and demand balance. One can model such a market by means of a stochastic process (Ch. III) with jumps (prototype: Poisson process).

With small trades, one can look at things at two different levels of detail. 'From a distance', prices seem to move continuously – so can be modelled by a stochastic process which is continuous (prototype: Brownian motion). But imagine a trader spending a trading day tracking the price movement of a heavily traded stock under normal market conditions. From a distance, price movement looks continuous, but close up, prices move by lots of little jumps – the effects of the individual small trades, and how they briefly affect the current balance between supply and demand. Such movement of prices by 'lots of little jumps' is called *jitter*.

Utility.

A pound is worth much more to a poor man than to a rich man. For ordinary people, a 10% increase in income might well give an extra 10% of satisfaction. So for small amounts x of money, we can think of utility as being the same as money. But to a billionaire, it would be hardly noticeable. To model this, one uses a *utility function*, $U(.)$, which measures how much *utility* – genuine use – money is to the economic agent in question: income x gives utility $U(x)$. The effect above is called the *Law of diminishing returns* (or *diminishing utility* – U is strictly increasing, but its graph bends below the line $y = x$, and indeed is typically bounded above.

Aside: Insurance. This underlies the insurance industry, as follows. An individual householder insures his house (against fire, subsidence etc.) not because he thinks it will burn or fall down, but because he knows he could not handle the consequences of it doing so if he were uninsured. He insures *expecting* to 'make a loss' – make no claim in return for his premium. By contrast, the insurance company expects to pay out on some claims, but to make money overall. The ability of the insurer and insured to agree on a premium level that they are both satisfied with comes from the fact that a claim is small for the company (on the straight part of *its* utility curve, where for them utility is money) but large for the householder (on the bent part of *his* utility curve – where utility and money are very different).

A guiding principle that is often used here is that each economic agent should seek to *maximize his expected utility.* This approach goes back to John Von Neumann and Oscar Morgenstern in 1947 (in their classic book *Theory of games and economic behaviour*, and earlier to F. P. Ramsey (1906-1930) in 1931 (posthumously).

Loss.

This is often looked at the other way round. One uses a *loss function* – which can usually be thought of as a negative of utility. One then seeks to *minimize one's expected loss.*

Arbitrage.

An *arbitrage* opportunity (see I.6) is the possibility of extracting riskless profit from the market. In an orderly market, this should not be possible – at least, to a first approximation. For, an arbitrage opportunity is 'free money'; arbitrageurs will take this, in unlimited quantities – until the person or institution being so exploited is driven from the market (bankrupt or otherwise). In view of this, we make the assumption that the market is *free of arbitrage* – is *arbitrage-free*, or has *no arbitrage*, NA.

Idealized markets.

Various assumptions are commonly made, in order to bring to bear the tools of mathematics on the broad field of economic/financial activity. All are useful, but valid to a first approximation only.

1. No arbitrage (NA).
2. No transaction costs or transaction taxes.
3. Same interest rates for borrowing and lending.
4. Unlimited liquidity (the ability to turn goods into money, and vice versa, at the currently quoted prices).
5. No limitations of scale.

Markets satisfying such assumptions will be called *perfect*, or *frictionless* – unrealistic in detail, but a useful first approximation in practice.

3. Brief history of Mathematical Finance

Mathematical Finance I: Markowitz and CAPM.

We deal with the history of put-call parity (I.7) below. It has ancient roots, but entered the textbooks around 1904.

Louis Bachelier (1870-1946) first put mathematics to work on finance in his 1900 thesis *Théorie de la spéculation*.[2]

Bachelier's thesis is also remarkable as he used *Brownian motion* as a model for the driving noise in the price of a risky asset. This was remarkable, as the relevant mathematics did not exist until 1923 (Wiener), and later (Itô, stochastic calculus, 1944).

Until 1952, finance was more an art than a science. This changed with the 1952 thesis of Harry Markowitz (1927–), which introduced modern *portfolio theory*. Markowitz gave us two key insights, both so 'obvious' that they are all around us now.

There is no point in investing in the stock market, which is risky, when one can instead invest risklessly by putting money in the bank, unless one expects the (rate of) *return* on the stock, μ, to be higher than the riskless return r. The riskiness of the stock is measured by a parameter, the *volatility* σ, which corresponds to the standard deviation (square root of the variance) in a model of the risky stock price as a stochastic process (Ch. III), while μ, r correspond to *means*, for risky and riskless assets respectively. Markowitz's first key insight is:

think of risk and return together, not separately.

This leads to *mean-variance analysis*.

[2] Mark Davis and Alison Etheridge: Louis Bachelier's *Theory of speculation*: The origins of modern finance, translated and with a commentary; foreword by Paul A. Samuelson. Princeton UP, 2006.

Next, the investor is free to choose which sector of the economy to invest in. He is investing in the face of uncertainty (or risk), and in each sector he chooses, prices may move against him. He should insure against this by holding a *balanced portfolio*, of assets from a number of different sectors, chosen so that they will tend to 'move against each other'. Then, 'what he loses on the swings he will gain on the roundabouts'. This tendency to move against each other is measured by *negative correlation* (the term comes from Statistics). Markowitz's second key insight is:

diversify, by holding a balanced portfolio with lots of negative correlation.

Markowitz's theory was developed during the 1960s, in the *capital asset pricing model* (CAPM – 'cap-emm'), of Sharpe, Lintner and Mossin (William Sharpe (1964), John Lintner (1965), Jan Mossin (1966); Jack Treynor (1961, 1962)). In CAPM, one looks at the excess of a particular stock over that of the market overall, and the risk (as measured by volatility), and seeks to obtain the maximum return for a given risk (or minimum risk for a given return), which will hold on the *efficient frontier*. The relevant mathematics involves Linear Regression in Statistics, and Linear Programming in Operational Research (OR).

Mathematical Finance II: Black, Scholes and Merton.

If one is contemplating buying a particular stock, intending to hold it for a year say, what one would love to know is the price in a year's time, compared with the price today (one should discount this, as above). If the (discounted) price goes up, one will be glad in a year's time that one bought; if it goes down, one will be sorry.

Suppose one's Fairy Godmother appeared, and gave one a piece of paper, which said that if one bought now, then in a year's time if one was glad one had done so one did buy, but if one was sorry, one didn't. Such pieces of paper do exist, and are called *options* – see Ch. IV, VI. Clearly such options are valuable: they may lead to a profit, but cannot lead to a loss.

Question: What is an option worth?

Note that unless one can it price options, they will not be traded (at least in any quantity) – as with anything else.

Before 1973, the conventional wisdom was that this question had no answer: it *could have no answer*, because the answer would necessarily depend on the economic agent's attitude to risk (that is, on his utility function, or loss function – see above). It turns out that this view is incorrect. Subject to the above assumptions of an idealized market (NA, etc.), one can price options, according to the famous *Black-Scholes formula* of 1973 (Ch. IV, VI – Fischer Black (1938-1995) and Myron Scholes (1941-)). They derived their formula by showing that the option price satisfied a partial differential equation (PDE), of hyperbolic type (a variant of the *heat equation*). In 1973 Robert Merton (1944-) gave a more direct approach. Meanwhile, 1973 was also the year when the first exchange for buying and selling options opened, the Chicago Board Options Exchange.

To see why options can be priced, one only needs to know that the standard options are (under our idealized assumptions) *redundant* financial assets: an option is equivalent to an appropriate combination of cash and stock. Knowing how much cash, how much stock and the current stock price, one can thus calculate the current option price by simple arithmetic.

In 1981, it was shown (by J. M. Harrison and S. R. Pliska) that the right mathematical machinery to use in this area involves a particular type of stochastic process – *martingales* – and a particular type of calculus, for stochastic processes – *Itô calculus* (Kiyosi Itô (1915-2008)); see Ch. VI.

The subject of Mathematical Finance is by now well-established, and rapidly growing in popularity in universities, in UK, US and elsewhere. This is because of its relevance to the needs of the financial sector (or financial services industry) in the City of London (also Edinburgh) within UK, New York in USA, Tokyo in Japan, Frankfurt in Germany, etc. This sector needs technical people with good skills in mathematics, statistics, numerics etc., as well as economic insight and financial awareness, problem-solving skills and ability to work in a team, etc. Such people are variously called financial engineers, quantitative analysts ('quants') or 'rocket scientists'.

Academically, the subject falls broadly in the interface between Economics on the one hand and Mathematics on the other. In Economics, much of the subject, again broadly speaking, relates to *how prices are determined* – by the interplay between supply and demand, etc. By contrast, here in this course we will usually take prices as given. Our task is to study how, starting from the given prices, we can price other things related to them (options, and other financial derivatives – see below), and guard our operations against unpredictable hazards (hedge – again, see below).

In this sense, Finance as a subject appears as a small – specialised, highly mathematical – part of Economics. Note that Finance here is not used quite in the traditional non-technical sense. In ordinary life, a financier is someone who decides whether or not to commit risk capital to a business proposition, brought to him by a businessman or entrepreneur who has some business idea but lacks the capital to implement it without borrowing money to finance it. Here, we may think of a fund manager – of an investment fund, say (such as manages my pension contributions, and will manage yours in the future). The range of investment opportunities open to our fund manager is almost limitless. They vary widely in (prospective) return and (apparent) risk. He will be judged by the financial performance of his fund, against his competitors on the one hand and the performance of an 'index' representing the market on the other. How should he proceed? Can Mathematics help him?

Risk is the key danger – the key concept even – in finance; risk reflects uncertainty; uncertainty reflects chance or probability. So it was clear that Probability Theory, a branch of Mathematics related to Statistics, had to be relevant here. Quite how was shown in 1981 by J. M. (Michael) Harrison (a probabilist) and David Kreps (an economist), who simplified and generalized the Black-Scholes-Merton theory by using the language of Probability Theory and Stochastic Processes – in particular, *martingales* (and Itô calculus, again). These developments – and what followed – constituted the 'second revolution in mathematical finance'. This is the subject-matter of this course. (We can cover the mathematics of the developments outlined above. Lots more has happened more recently, and that's where a lot of the interest is focussed today. But this goes beyond what one can cover in an undergraduate course.) On the mathematical side: you will learn a lot about stochastic processes, martingales and Itô calculus, and see them put to use on financial problems. On the practical side: the best proof of the relevance and usefulness of these ideas is the explosive growth in volumes of trades in financial derivatives over the last thirty years, and the corresponding explosive growth in employment opportunities (and salaries!) for those who understand what is going on.

As with everything else in life, triumph and disaster can always happen, and one has to use common sense. Triumph: Scholes and Merton were awarded the Nobel Prize for Economics in 1997 (Black died in 1995, and the prize cannot be awarded posthumously). Disaster: Scholes and Merton were on the board of the hedge fund Long Term Capital Management, which ignominiously collapsed with enormous losses in 1998. Pushing a good theory too far – beyond all sensible limits – is asking for trouble, even if one invented the theory and got the Nobel Prize for it, and if one asks for trouble, one can expect to get it.

4. Markets and Options.

Markets.

This course is about the mathematics needed to model *financial markets*. These are of several types:

Stock markets [New York, London, ...], dealing in stocks/shares/equities, etc.,
Bond markets, dealing in government bonds (gilts, ...),
Currency or foreign exchange ('forex') markets,
Futures and *options markets*, dealing in financial instruments derived from the above - *financial derivatives* such as *options* of various types.

Options.

Economic activity, and trading, involves *risk*. One may have to, or choose to, make a judgement involving committing funds ('taking a position') based on prediction of the future in the presence of uncertainty. With hindsight, one might or might not regret taking that position. An *option* is a financial instrument giving one the *right but not the obligation* to make a specified transaction at (or by) a specified date at a specified price. Whether or not the option will be exercised depends on (is contingent on) the uncertain future, so is also known as a *contingent claim*.

Types of option.

Call options give one the right (but not – without further comment now – the obligation) to *buy*.

Put options give one the right to *sell*.

European options give one the right to buy/sell *on* the specified date, the *expiry* date, when the option *expires* or matures.

American options give one the right to buy/sell *at any time* prior to or at expiry. Thus:

European options: exercise *at* expiry,
American options: exercise *by* expiry.

Note. The terms European, American (Asian, Bermudan, Russian, ...) refer only to the type of option, and no longer bear any relation to the area in the name. Most options traded worldwide these days are American.

History. As discussed in §1, over-the-counter (OTC) options were long ago negotiated by a broker between a buyer and a seller. Then in 1973 (the year of the Black-Scholes formula, perhaps the central result of the course), the Chicago Board Options Exchange (CBOE) began trading in options on some stocks. Since then, the growth of options has been explosive. Options are now traded on all the major world exchanges, in enormous volumes. Often, the market in derivatives is *much larger* than the market in the underlying assets – an important source of instability in financial markets.

The simplest call and put options are now so standard they are called *vanilla* options. Many kinds of options now exist, including so-called *exotic* options. Types include:
Asian options, which depend on the *average* price over a period,
Russian options, or other *lookback* options, which depend on the *maximum* or *minimum* price over a period,
Barrier options, which depend on some price level being attained or not.
Real options (also called *investment options*). These are 'options' available to the management of a company considering whether or when to commit capital (usually both irreversibly and riskily) to some investment project. Waiting may be valuable, as one can gather more information.
Terminology. The asset to which the option refers is called the *underlying asset* or the *underlying*. The price at which the transaction to buy/sell the underlying, on/by the expiry date (if exercised), is called the *exercise price* or *strike price*. We shall usually use K for the strike price, time $t = 0$ for the initial time (when the contract between the buyer and the seller of the option is struck), time $t = T$ for the expiry or final time.

Consider, say, a European *call* option, with strike price K; write S_t for the value (or price) of the underlying at time t. If $S_T > K$, the option is *in the money*: the holder will/should *exercise* the option, obtaining an asset worth S_T ($> K$) for K. He can immediately sell the asset for S_T, making a *profit* of $S_T - K$ (> 0).
If $S_T = K$, the option is said to be *at the money*.
If $S_T < K$, the option is *out of the money*, and should not be exercised. It is worthless, and is thrown away.

The *pay-off* from the option is thus

$$S_T - K \text{ if } S_T > K, \quad 0 \quad \text{otherwise},$$

which may be written more briefly as

$$max(S_T - K, 0) \text{ or } (S_T - K)_+$$

$(x_+ := max(x, 0), x_- := -min(x, 0); x = x_+ - x_-, |x| = x_+ + x_-)$.
Similarly, the payoff from a *put* option is

$$K - S_T \text{ if } S_T \leq K, \qquad 0 \text{ if } S_T > K,$$

or $(K - S_T)_+$.

Option pricing. The fundamental problem in the mathematics of options is that of *option pricing*. The modern theory began with the *Black-Scholes formula* for pricing European options in 1973. We shall deal with the Black-Scholes theory, and cover the pricing of European options in full. We also discuss American options: these are harder, and lack explicit formulae such as the Black-Scholes formula; consequently, one needs to evaluate them numerically. The pricing of Asian options is even harder and is still topical at research level.

Perfect Markets. For simplicity, we shall confine ourselves to option pricing in the simplest (idealised) case, of a *perfect*, or *frictionless*, market. First, there are no *transaction costs* (one can include transaction costs in the theory, but this is considerably harder). Similarly, we assume that interest rates for borrowing and for lending are the same (which is unrealistic, as banks make their money on the difference), and also that all traders have access to the same – perfect – information about the past history of price movements, but have no foreknowledge of price-sensitive information (i.e. no insider trading). We shall assume no restrictions on *liquidity* – that is, one can buy or sell unlimited quantities of stock at the currently quoted price. That is, our economic agents are *price takers* and not *price makers*. (This comes back to §1 on the relationship between Economics and Finance. In practice, big trades do move markets. Also, in a crisis, no-one wants to trade, and liquidity dries up – basically, this is what did for LTCM.) In practice, very small trades are not economic (the stockbroker may only deal in units of reasonable size, etc.). We shall ignore all these complications for the sake of simplicity.

References. For completeness, classic papers include:

[BS] BLACK, F. & SCHOLES, M. (1973): The pricing of options and corporate liabilities, *J. Political Economy* **81**, 637-659,

[HK] HARRISON, J. M. & KREPS, D. M. (1979): Martingales and arbitrage in multiperiod securities markets. *J. Econ. Th.* **20**, 381-408,

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[SKKM] SHIRYAEV, A. N, KABANOV, Yu. M., KRAMKOV, O. D. & MELNIKOV, A. V. (1995): Towards the theory of pricing of options of both European and American types. I: Discrete time, II: Continuous time. *Theory of Probability and Applications* **39**.1, 14-60, 61-102 [and further papers in the same issue of TPA].