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The Black-Scholes Model (continued)

The discounted value process is

$$\tilde{V}_t(H) = e^{-rt} V_t(H)$$

and the interest rate is r. So

$$d\tilde{V}_t(H) = -re^{-rt} dt.V_t(H) + e^{-rt} dV_t(H)$$

(since e^{-rt} has finite variation, this follows from the integration-by-parts formula

$$d(XY)_t = X_t dY_t + Y_t dX_t + \frac{1}{2} d\langle X, Y\rangle_t$$

– the quadratic covariation of a finite-variation term with any term is zero, so there is no extra term)

$$\begin{aligned} &= -re^{-rt} H_t.S_t dt + e^{-rt} H_t.dS_t \\ &= H_t.(-re^{-rt} S_t dt + e^{-rt} dS_t) \\ &= H_t.d\tilde{S}_t \end{aligned}$$

($\tilde{S}_t = e^{-rt} S_t$, so $d\tilde{S}_t = -re^{-rt} S_t dt + e^{-rt} dS_t$ as above): for H self-financing,

$$dV_t(H) = H_t.dS_t, \qquad d\tilde{V}_t(H) = H_t.d\tilde{S}_t,$$

$$V_t(H) = V_0(H) + \int_0^t H_s dS_s, \qquad \tilde{V}_t(H) = \tilde{V}_0(H) + \int_0^t H_s d\tilde{S}_s.$$

Now write $U_t^i := H_t^i S_t^i / V_t(H) = H_t^i S_t^i / \Sigma_j H_t^j S_t^j$ for the *proportion* of the value of the portfolio held in asset $i = 0, 1, \cdots, d$. Then $\Sigma U_t^i = 1$, and $U_t = (U_t^0, \cdots, U_t^d)$ is called the *relative portfolio*. For H self-financing,

$$dV_t = H_t.dS_t = \Sigma H_t^i dS_t^i = V_t \Sigma \frac{H_t^i S_t^i}{V_t}.\frac{dS_t^i}{S_t^i},$$

or

$$dV_t = V_t \Sigma U_t^i dS_t^i / S_t^i.$$

Dividing through by V_t, this says that the return dV_t/V_t is the weighted average of the returns dS_t^i/S_t^i on the assets, weighted according to their proportions U_t^i in the portfolio.

Note. Having set up this notation (that of [HP]) – in order to be able if we wish to have a basket of assets in our portfolio – we now prefer – for simplicity – to specialise back to the simplest case, that of one risky asset. Thus we will now take $d = 1$ until further notice.

Arbitrage. This is defined as in discrete time: an admissible ($V_t(H) \geq 0$ for all t) self-financing strategy H is an *arbitrage* (strategy, or opportunity) if

$$V_0(H) = 0, \qquad V_T(H) > 0 \qquad \text{with positive } P\text{-probability.}$$

The market is *viable*, or *arbitrage-free*, or NA, if there are no arbitrage opportunities.

We see first that if the value-process V satisfies the SDE

$$dV_t(H) = K(t)V_t(H)dt$$

– that is, if there is no driving Wiener (or noise) term – then $K(t) = r$, the short rate of interest. For, if $K(t) > r$, we can *borrow* money from the bank at rate r and *buy* the portfolio. The value grows at rate $K(t)$, our debt grows at rate r, so our net profit grows at rate $K(t) - r > 0$ – an arbitrage. Similarly, if $K(t) < r$, we can *invest* money in the bank and *sell the portfolio short*. Our net profit grows at rate $r - K(t) > 0$, risklessly – again an arbitrage. We have proved the

Proposition. In an arbitrage-free (NA) market, a portfolio whose value process has no driving Wiener term in its dynamics must have return rate r, the short rate of interest.

We restrict attention to arbitrage-free (viable) markets from now on.

We now consider tradeable derivatives, whose price at expiry depends only on $S(T)$ (the final value of the stock) – $h(S(T))$, say, and whose price process Π_t depends on the asset price S_t in a smooth way: for some smooth function F,

$$\Pi_t := F(t, S_t).$$

The dynamics of the riskless and risky assets are

$$dB_t = rB_t dt, \qquad dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ, σ may depend on both t and S_t:

$$\mu = \mu(t, S_t), \qquad \sigma = \sigma(t, S_t).$$

By Itô's Lemma,

$$d\Pi_t = F_1 dt + F_2 dS_t + \frac{1}{2} F_{22} (dS_t)^2$$

(since t has finite variation, the F_{11}- and F_{12}-terms are absent as $(dt)^2$ and $dtdS_t$ are negligible with respect to the terms retained)

$$= F_1 dt + F_2 (\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2} F_{22} (\sigma S_t dW_t)^2$$

(since the contribution of the finite-variation term in dt is negligible in the second differential, as above)

$$= (F_1 + \mu S_t F_2 + \frac{1}{2} \sigma^2 {S_t}^2 F_{22}) dt + \sigma S_t F_2 dW_t$$

(as $(dW_t)^2 = dt$). Now $\Pi = F$, so

$$d\Pi_t = \Pi_t (\mu_\Pi(t) dt + \sigma_\Pi(t) dW_t),$$

where

$$\mu_\Pi(t) := (F_1 + \mu S_t F_2 + \frac{1}{2} \sigma^2 S_t^2 F_{22})/F, \qquad \sigma_\Pi(t) := \sigma S_t F_2 / F.$$

Now form a portfolio based on two assets: the underlying stock and the derivative asset. Let the relative portfolio in stock S and derivative Π be (U_t^S, U_t^Π). Then the dynamics for the value V of the portfolio are given by

$$\begin{aligned} \frac{dV_t}{V_t} &= U_t^S \frac{dS_t}{S_t} + U_t^\Pi \frac{d\Pi_t}{\Pi_t} \\ &= U_t^S (\mu dt + \sigma dW_t) + U_t^\Pi (\mu_\Pi dt + \sigma_\Pi dW_t) \\ &= (U_t^S \mu + U_t^\Pi \mu_\Pi) dt + (U_t^S \sigma + U_t^\Pi \sigma_\Pi) dW_t, \end{aligned}$$

by above. Now both brackets are linear in U^S, U^Π, and $U^S + U^\Pi = 1$ as proportions sum to 1. This is one linear equation in the two unknowns U^S, U^Π, and we can obtain a second one by eliminating the driving Wiener term in the dynamics of V – for then, the portfolio is *riskless*, so must have return r by the Proposition, to avoid arbitrage. We thus solve the two equations

$$\begin{aligned} U^S + U^\Pi &= 1 \\ U^S \sigma + U^\Pi \sigma_\Pi &= 0. \end{aligned}$$

The solution of the two equations above is

$$U^{\Pi} = \frac{\sigma}{\sigma - \sigma_{\Pi}}, \qquad U^{S} = \frac{-\sigma_{\Pi}}{\sigma - \sigma_{\Pi}},$$

which as $\sigma_{\Pi} = \sigma S F_2 / F$ gives the portfolio explicitly as

$$U^{\Pi} = \frac{F}{F - SF_2}, \qquad U^{S} = \frac{-SF_2}{F - SF_2}.$$

With this choice of relative portfolio, the dynamics of V are given by

$$dV_t/V = (U_t^S \mu + U_t^{\Pi} \mu_{\Pi})dt,$$

which has no driving Wiener term. So, no arbitrage as above implies that the return rate is the short interest rate r:

$$U_t^S \mu + U_t^{\Pi} \mu_{\Pi} = r.$$

Now substitute the values (obtained above)

$$\mu_{\Pi} = \frac{F + \mu S F_2 + \frac{1}{2}\sigma^2 S^2 F_{22}}{F}, \qquad U^S = \frac{-SF_2}{F - SF_2}, \qquad U^{\Pi} = \frac{F}{F - SF_2}$$

in this no-arbitrage relation:

$$\frac{-SF_2}{F - SF_2}.\mu + \frac{F}{F - SF_2}.\frac{F_1 + \mu S F_2 + \frac{1}{2}\sigma^2 F_{22}}{F} = r.$$

So

$$-SF_2\mu + F_1 + \mu S F_2 + \frac{1}{2}\sigma^2 S^2 F_{22} = rF - rSF_2,$$

giving

$$F_1 + rSF_2 + \frac{1}{2}\sigma^2 S^2 F_{22} - rF = 0. \qquad (BS)$$

This is the celebrated *Black-Scholes partial differential equation* (PDE) of 1973, and proves one of the central results of the subject:

Theorem (Black-Scholes PDE). In a market with one riskless asset B_t and one risky asset S_t, with short interest-rate r and dynamics

$$\begin{aligned} dB_t &= rB_t dt, \\ dS_t &= \mu(t, S_t) S_t dt + \sigma(t, S_t) S_t dW_t, \end{aligned}$$

let a contingent claim be tradeable, with price $h(S_T)$ at expiry T and price process $\Pi_t := F(t, S_t)$ for some smooth function F. Then the only pricing function F which does not admit arbitrage is the solution to the Black-Scholes PDE with boundary condition:

$$F_1(t,x) + rxF_2(t,x) + \frac{1}{2}x^2\sigma^2(t,x)F_{22}(t,x) - rF(t,x) = 0, \qquad (BS)$$

$$F(T,x) = h(x). \qquad (BC)$$

Corollary. The no-arbitrage price of the derivative does not depend on the mean return $\mu(t,.)$ of the underlying asset, only on its *volatility* $\sigma(t,.)$ and the short interest-rate.

The Black-Scholes PDE may be solved analytically, or numerically. We give an alternative probabilistic approach below.
Note: Partial Differential Equations (PDEs). The most important PDEs encountered in Mathematics or Physics (or Finance!) are *linear* PDEs of *second order* (involving partial derivatives of first or second order only). These may be classified, in a way analogous to the classification of *conics* or conic sections (whose equations are algebraic of second order), into three broad categories:
Elliptic PDEs – prototype, *Laplace's equation*

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$$

(or *Poisson's equation*, with $4\pi\rho$ on RHS) in electromagnetism or potential theory;
Parabolic PDEs – prototype, the *heat equation*

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = \kappa^{-1}\partial u/\partial t;$$

Hyperbolic PDEs – prototype, the *wave equation*

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = c^{-2}\partial u/\partial t^2.$$

The Black-Scholes PDE is parabolic, and can be transformed into the heat equation, whose solution can be written down in terms of an integral and the *heat kernel*. This is the same as the probabilistic solution obtained below.
Note. 1. Black and Scholes were classically trained applied mathematicians. When they derived their PDE, they recognised it as parabolic. After some months' work, they were able to transform it into the heat equation. The

solution to this is known classically.[1] On transforming back, they obtained the Black-Scholes formula.

The Black-Scholes formula transformed the financial world. Before it (see Ch. I), the expert view was that asking what an option is worth was (in effect) a silly question: the answer would necessarily depend on the attitude to risk of the individual considering buying the option. It turned out that – at least approximately (i.e., subject to the restrictions to perfect – frictionless – markets, including No Arbitrage – an over-simplification of reality) there *is* an option value. One can see this in one's head, without doing any mathematics, if one knows that the Black-Scholes market is *complete*. So, every contingent claim (option, etc.) can be *replicated*, in terms of a suitable combination of cash and stock. Anyone can price this:
(i) count the cash, and count the stock;
(ii) look up the current stock price;
(iii) do the arithmetic.
2. The programmable pocket calculator was becoming available around this time. Every trader immediately got one, and programmed it, so that he could price an option (using the Black-Scholes model!) in real time, from market data.
3. The missing quantity in the Black-Scholes formula is the *volatility*, σ. But, the price is continuous and strictly increasing in σ (options like volatility!). So there is *exactly one* value of σ that gives the price at which options are being currently traded. The conclusion is that this is the value that the market currently judges σ to be. This is the value (called the *implied volatility* that traders use.
4. Because the Black-Scholes model is the benchmark model of mathematical finance, and gives a value for σ at the push of a button, it is widely used.
5. This is *despite* the fact that no one actually believes the Black-Scholes model! It gives at best an over-simplified approximation to reality. Indeed, Fischer Black himself famously once wrote a paper called *The holes in Black-Scholes*.

[1]See e.g. the link to MPC2 (Mathematics and Physics for Chemists, Year 2) on my website, Weeks 4, 9. The solution is in terms of *Green functions*. The Green function for (fundamental solution of) the heat equation has the form of a normal density. This reflects the close link between the mathematics of the heat equation (J. Fourier (1768-1830) in 1807; *Théorie analytique de la chaleur* in 1822) and the mathematics of Brownian motion, which as we have seen belongs to the 20th Century. The link was made by S. Kakutani in 1944, and involves potential theory.

6. This is an interesting example of theory and practice interacting!
7. Black and Scholes has considerable difficulty in getting their paper published! It was ahead of its time. When published, and its importance understood, it changed its times.

§3. The Feynman-Kac Formula, Risk-Neutral Valuation and the Continuous Black-Scholes Formula

Suppose we consider a SDE, with initial condition (IC), of the form

$$dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s \qquad (t \leq s \leq T), \tag{SDE}$$

$$X_t = x. \tag{IC}$$

For suitably well-behaved functions μ, σ, this SDE will have a unique solution $X = (X_s : t \leq s \leq T)$, a diffusion. We refer for details on solutions of SDEs and diffusions to an advanced text such as [RW2], [RY], [KS].

Taking existence of a unique solution for granted for the moment, consider a smooth function $F(s, X_s)$ of it. By Itô's Lemma,

$$dF = F_1 ds + F_2 dX + \frac{1}{2}F_{22}(dX)^2,$$

and as $(dX)^2 = (\mu ds + \sigma dW_s)^2 = \sigma^2(dW_s)^2 = \sigma^2 ds$, this is

$$dF = F_1 ds + F_2(\mu ds + \sigma dW_s) + \frac{1}{2}\sigma^2 F_{22} ds = (F_1 + \mu F_2 + \frac{1}{2}\sigma^2 F_{22})ds + \sigma F_2 dW_s. \tag{$*$}$$

Now suppose that F satisfies the PDE, with boundary condition (BC),

$$F_1(t, x) + \mu(t, x)F_2(t, x) + \frac{1}{2}\sigma^2 F_{22}(t, x) = 0 \tag{PDE}$$

$$F(T, x) = h(x). \tag{BC}$$

Then the above expression for dF gives

$$dF = \sigma F_2 dW_S,$$

which can be written in stochastic-integral rather than stochastic-differential form as

$$(T, X_T) = F(t, X_t) + \int_t^T \sigma(s, X_s)F_2(s, X_s)dW_s.$$

The stochastic integral on the right is a martingale, so has constant expectation, which must be 0 as it starts at 0. Recalling that $X_t = x$, and writing $E_{t,x}$ for expectation with value x and starting-time t,

$$F(t, x) = E_{t,x}F(T, X_T).$$

Writing the price at expiry T as $h(X_T)$ as before, this gives

$$F(t, x) = E_{t,x}h(X_T) :$$

Theorem (Feynman-Kac Formula). The solution $F = F(t, x)$ to the PDE

$$F_1(t, x) + \mu(t, x)F_2(t, x) + \frac{1}{2}\sigma^2(t, x)F_{22}(t, x) = 0 \qquad (PDE)$$

with final condition $F(T, x) = h(x)$ has the stochastic representation

$$F(t, x) = E_{t,x}h(X_T), \qquad (FK)$$

where X satisfies the SDE

$$dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s \qquad (t \leq s \leq T) \qquad (SDE)$$

with initial condition $X_t = x$.

Now replace $\mu(t, x)$ by rx, $\sigma(t, x)$ by σx in the Feynman-Kac formula above. The SDE becomes

$$dX_s = rX_s ds + \sigma X_s dW_s \qquad (**)$$

– the same as for a risky asset with mean return-rate r (the short interest-rate for a riskless asset) in place of μ (which disappeared in the Black-Scholes result). The PDE becomes

$$F_1 + rxF_2 + \frac{1}{2}\sigma^2 x^2 F_{22} = 0,$$

which is the Black-Scholes PDE except that there the LHS above is rF rather than 0. So we can study solutions to the Black-Scholes PDE by Feynman-Kac methods, by returning to the proof of the Feynman-Kac formula and replacing $F_1 + rxF_2 + \frac{1}{2}x^2F_{22}$ in $(*)$ by rF:

$$dF = rFds + \sigma F_2 dW_s, \qquad F(T, s) = h(s).$$

We can eliminate the first term on the right by discounting at rate r: write $G(s, X_s) := e^{-rs}F(s, X_s)$ for the discounted price process. Then as before,

$$dG = -re^{-rs}Fds + e^{-rs}dF = e^{-rs}(dF - rFds) = e^{-rs}.\sigma F_2 dW.$$

Then integrating, G is a stochastic integral, so a martingale: *the discounted price process* $G(s, X_s) = e^{-rs}F(s, X_s)$ *is a martingale*, under the measure P^* giving the dynamics in $(**)$. This is the measure P we started with, *except* that μ has been changed to r. Thus, G has constant P^*-expectation:

$$E^*_{t,x}G(t, X_t) = E^*_{t,x}e^{-rt}F(t, X_t) = e^{-rt}F(t, x) = E^*_{T,x}e^{-rT}F(T, X_T) = e^{-rT}h(X_T):$$

Theorem (Risk-Neutral Valuation Formula). The no-arbitrage price of the claim $h(S_T)$ is given by

$$F(t, x) = e^{-r(T-t)}E^*_{t,x}h(S_T),$$

where $S_t = x$ is the asset price at time t and P^* is the measure under which the asset price dynamics are given by

$$dS_t = rS_t dt + \sigma(t, S_t)S_t dW_t.$$

Corollary. In the Black-Scholes model above, the arbitrage-free price does not depend on the mean return rate μ of the underlying asset.

Comments.

1. *Risk-neutral measure.* We call P^* the *risk-neutral* probability measure. It is equivalent to P (by Girsanov's Theorem – the change-of-measure result, which deals with change of drift in SDEs – see Week 11), and is a martingale measure (as the discounted asset prices are P^*-martingales, by above), i.e. P^* (or Q) is the *equivalent martingale measure (EMM).*
2. *Fundamental Theorem of Asset Pricing.* The above continuous-time result may be summarised just as the Fundamental Theorem of Asset Pricing in discrete time: to get the no-arbitrage price of a contingent claim, take the discounted expected value under the equivalent mg – or risk-neutral – measure.
3. *Completeness.* In discrete time, we saw that absence of arbitrage corresponded to *existence* of risk-neutral measures, completeness to *uniqueness*. We have obtained existence and uniqueness here (and so completeness), by appealing to existence and uniqueness theorems for PDEs (which we have not

proved!). A more probabilistic route is to use Girsanov's Theorem (Week 11) instead. Completeness questions then become questions on representation theorems for Brownian martingales (Week 11). As usual, there is a choice of routes to the major results – in this case, a trade-off between analysis (PDEs) and probability (Girsanov's Theorem and the Representation Theorem for Brownian Martingales, Week 11).

4. Calculation. When solutions have to be found numerically (as is the case in general - though not for some important special cases such as European call options, considered below), we again have a choice of

(i) analytic methods: numerical solution of a PDE,

(ii) probabilistic methods: evaluation, by the Risk-Neutral Valuation Formula, of an expectation.

A comparison of convenience between these two methods depends on one's experience of numerical computation and the software available. However, in the simplest case considered here, the probabilistic problem involves a one-dimensional integral, while the analytic problem is two-dimensional (involves a two-variable PDE: one variable would give an ODE!). So on dimensional grounds, and because of the probabilistic content of this course, we will generally prefer the probabilistic approach.

5. The Feynman-Kac formula. It is interesting to note that the Feynman-Kac formula originates in an entirely different context, namely quantum physics. In the late 1940s, the physicist Richard Feynman developed his path-integral approach to quantum mechanics, leading to his work (with Schwinger, Tomonaga and Dyson) on QED (quantum electrodynamics). Feynman's approach was non-rigorous; Mark Kac, an analyst and probabilist with an excellent background in PDE, produced a rigorous version which led to the approach above.

6. The Sharpe ratio. There is no point in investing in a risky asset with mean return rate μ, when cash is a riskless asset with return rate r, unless $\mu > r$. The excess return $\mu - r$ is compared with the risk, as measured by the volatility σ via the *Sharpe ratio*

$$\lambda := (\mu - r)/\sigma,$$

also known as the *market price of risk*.

Now the process specified under P^* by the dynamics $(**)$ is our old friend geometric Brownian motion, $GBM(r, \sigma)$. Thus if S_t has P^*-dynamics

$$dS_t = rS_t dt + \sigma S_t dW_t, \qquad S_t = s,$$

with W a P^*-Brownian motion, then we can write S_T explicitly as

$$S_T = s\exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)\}.$$

Now $W_T - W_t$ is normal $N(0, T-t)$, so $(W_T - W_t)/\sqrt{T-t} =: Z \sim N(0,1)$:

$$S_T = s\exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma Z\sqrt{T-t}\}, \qquad Z \sim N(0,1).$$

So by the Risk-Neutral Valuation Formula, the pricing formula is

$$F(t,x) = e^{-r(T-t)}\int_{-\infty}^{\infty} h(s\exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(T-t)^{\frac{1}{2}}x\}).\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx.$$

For a general payoff function h, there is no explicit formula for the integral, which has to be evaluated numerically. But we can evaluate the integral for the basic case of a European call option with strike-price K:

$$h(s) = (s-K)^+.$$

Then

$$F(t,x) = e^{-r(T-t)}\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}[s\exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(T-t)^{\frac{1}{2}}x\} - K]_+dx.$$

We have already evaluated integrals of this type in Chapter IV, where we obtained the Black-Scholes formula from the binomial model by a passage to the limit. Completing the square in the exponential as before gives the

Continuous Black-Scholes Formula.

$$F(t,s) = s\Phi(d_+) - e^{-r(T-t)}K\Phi(d_-),$$

where (writing Φ for the standard normal distribution function)

$$d_\pm := [\log(s/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{T-t}.$$

Note.
1 *Delta.* Writing Π as before for the option price, S for the asset price, the partial derivative of Π w.r.t. S is called the *delta* of the option:

$$\Delta := \partial\Pi/\partial S.$$

The Black-Scholes analysis above works by balancing a holding of option against stock so as to make the portfolio instantaneously riskless; the method is known as *delta-hedging*. Note that the portfolio, being only instantaneously riskless, can only be kept so by continuous re-balancing. This involves infinite amounts of trading, an admitted idealisation. In particular, delta-hedging is vulnerable to *transaction costs*, present in real markets, and in more realistic – but more complicated – models; see §6.3 below.
2 *The Greeks*. Other important characteristics, traditionally labelled by Greek letters as above, are:
(i) *gamma*: $\Gamma := \partial^2\Pi/\partial S^2$ (the 'curvature'),
(ii) *theta*: $\Theta := \partial\Pi/\partial t$;
(iii) *rho*: $\rho := \partial\Pi/\partial r$;
(iv) *vega*, the partial derivative w.r.t. volatility σ: $\partial\Pi/\partial\sigma$.

As in Problems 7: *vega* > 0 ('options like volatility'); delta for calls and puts satisfies $\Delta_C \in (0,1)$ (delta goes up with the price, but less steeply than the price, so call options are worthwhile as hedges against price increases); $\Delta_P \in (-1,0)$ (delta goes down with the price, but less steeply than the price, so put options are worthwhile as hedges against price decreases).
3. To put the basic case (μ and σ constant) in a nutshell:
(i). Dynamics are given by GBM, $dS_t = \mu S dt + \sigma S dW_t$.
(ii). Discount: $d\tilde{S}_y = (\mu - r)\tilde{S}dt + \sigma\tilde{S}dW_t$.
(iii). Use Girsanov's Theorem (Week 11) to change μ to r: under P^*, $d\tilde{S}_t = \sigma\tilde{S}dW_t$.
(iv). Integrate: the RHS gives a P^*-martingale, so has constant E^*-expectation.
4. One often has a choice between discrete and continuous time. For discrete time, we have proved everything; for continuous time, we have had to quote the hard proofs. Note that in continuous time we can use calculus – PDEs, SDEs etc. In discrete time we use instead the *calculus of finite differences*.
5. The calculus of finite differences is very similar to ordinary calculus (old-fashioned name: the *infinitesimal calculus* – thus the opposite of finite here is infinitesimal, not infinite!). It is in some ways harder. For instance: you all know integration by parts (partial integration) backwards. The discrete analogue – partial summation, or Abel's lemma – may be less familiar.

The calculus of finite differences used to be taught for use in e.g. interpolation (how to use information in mathematical tables to 'fill in missing values'). This is now done by computer subroutines – *but*, computers work discretely (with differences rather than derivatives), so the subject is still alive and well.