liv6.tex Week 6: 3.3.2014

## §3. Complete Markets: Uniqueness of EMMs.

A contingent claim (option, etc.) can be defined by its *payoff* function, $h$ say, which should be non-negative (options confer rights, not obligations, so negative values are avoided by not exercising the option), and $\mathcal{F}_N$-measurable (so that we know how to evaluate $h$ at the terminal time $N$).

**Definition.** A contingent claim defined by the payoff function $h$ is *attainable* if there is an admissible strategy worth (i.e., replicating) $h$ at time $N$. A market is *complete* if every contingent claim is attainable.

**Theorem (complete iff EMM unique)**. A viable market is complete iff there exists a unique probability measure $P^*$ equivalent to $P$ under which discounted asset prices are martingales – that is, iff equivalent martingale measures are unique.

*Proof.* $\Rightarrow$: Assume viability and completeness. Then for any $\mathcal{F}_N$-measurable random variable $h \geq 0$, there exists an admissible (so self-financing) strategy $H$ replicating $h$: $h = V_N(H)$. As $H$ is self-financing, by §1

$$h/S_N^0 = \tilde{V}_N(H) = V_0(H) + \Sigma_1^N H_j.\Delta\tilde{S}_j.$$

We know by the Theorem of §2 that an equivalent martingale measure $P^*$ exists; we have to prove uniqueness. So, let $P_1, P_2$ be two such equivalent martingale measures. For $i = 1, 2$, $(\tilde{V}_n(H))_{n=0}^N$ is a $P_i$-martingale. So,

$$E_i(\tilde{V}_N(H)) = E_i(V_0(H)) = V_0(H),$$

since the value at time zero is non-random ($\mathcal{F}_0 = \{\emptyset, \Omega\}$). So

$$E_1(h/S_N^0) = E_2(h/S_N^0).$$

Since $h$ is arbitrary, $E_1, E_2$ have to agree on integrating all non-negative integrands. Taking negatives and using linearity: they have to agree on non-positive integrands also. Splitting an arbitrary integrand into its positive and negative parts: they have to agree on all integrands. Now $E_i$ is expectation (i.e., integration) with respect to the measure $P_i$, and measures that agree on integrating all integrands must coincide. So $P_1 = P_2$. //

Before proving the converse, we prove a lemma. Recall that an admissible strategy is a self-financing strategy with all values non-negative. The Lemma shows that the non-negativity of contingent claims extends to all values of any self-financing strategy replicating it – in other words, this gives equivalence of admissible and self-financing replicating strategies.

**Lemma**. In a viable market, any attainable $h$ (i.e., any $h$ that can be replicated by an admissible strategy $H$) can also be replicated by a self-financing strategy $H$.

*Proof.* If $H$ is self-financing and $P^*$ is an equivalent martingale measure under which discounted prices $\tilde{S}$ are $P^*$-martingales (such $P^*$ exist by viability and the Theorem of §2), $\tilde{V}_n(H)$ is also a $P^*$-martingale, being the martingale transform of $\tilde{S}$ by $H$ (see §1). So

$$\tilde{V}_n(H) = E^*(\tilde{V}_N(H)|\mathcal{F}_n) \qquad (n = 0, 1, \cdots, N).$$

If $H$ replicates $h$, $V_N(H) = h \geq 0$, so discounting, $\tilde{V}_N(H) \geq 0$, so the above equation gives $\tilde{V}_n(H) \geq 0$ for each $n$. Thus *all* the values at each time $n$ are non-negative – not just the final value at time $N$ – so $H$ is admissible. //

*Proof of the Theorem (continued).* $\Leftarrow$ *(not examinable)*: Assume the market is viable but incomplete: then there exists a non-attainable $h \geq 0$. By the Lemma, we may confine attention to self-financing strategies $H$ (which will then automatically be admissible). By the Proposition of §1, we may confine attention to the risky assets $S^1, \cdots, S^d$, as these suffice to tell us how to handle the bank account $S^0$.

Call $\tilde{\mathcal{V}}$ the set of random variables of the form

$$U_0 + \Sigma_1^N H_n . \Delta \tilde{S}_n$$

with $U_0$ $\mathcal{F}_0$-measurable (i.e. deterministic) and $((H_n^1, \cdots, H_n^d))_{n=0}^N$ predictable; this is a vector space. Then by above, the discounted value $h/S_N^0$ does not belong to $\tilde{\mathcal{V}}$, so $\tilde{\mathcal{V}}$ is a *proper* subspace of the vector space $\mathbf{R}^\Omega$ of all random variables on $\Omega$. Let $P^*$ be a probability measure equivalent to $P$ under which discounted prices are martingales (such $P^*$ exist by viability, by the Theorem of §2). Define the scalar product

$$(X, Y) \to E^*(XY)$$

on random variables on $\Omega$. Since $\tilde{\mathcal{V}}$ is a proper subspace, by Gram-Schmidt orthogonalisation there exists a non-zero random variable $X$ orthogonal to $\tilde{\mathcal{V}}$. That is,

$$E^*(X) = 0.$$

Write $\|X\|_\infty := \max\{|X(\omega)| : \omega \in \Omega\}$, and define $P^{**}$ by

$$P^{**}(\{\omega\}) = \Big(1 + \frac{X(\omega)}{2\|X\|_\infty}\Big) P^*(\{\omega\}).$$

By construction, $P^{**}$ is equivalent to $P^*$ (same null-sets - actually, as $P^* \sim P$ and $P$ has no non-empty null-sets, neither do $P^*, P^{**}$). As $X$ is non-zero, $P^{**}$ and $P^*$ are *different.* Now

$$\begin{aligned} E^{**}(\Sigma_1^N H_n.\Delta\tilde{S}_n) &= \Sigma_\omega P^{**}(\omega)\Big(\Sigma_1^N H_n.\Delta\tilde{S}_n\Big)(\omega) \\ &= \Sigma_\omega\Big(1 + \frac{X(\omega)}{2\|X\|_\infty}\Big) P^*(\omega)\Big(\Sigma_1^N H_n.\Delta\tilde{S}_n\Big)(\omega). \end{aligned}$$

The '1' term on the right gives $E^*(\Sigma_1^N H_n.\Delta\tilde{S}_n)$, which is zero since this is a martingale transform of the $E^*$-martingale $\tilde{S}_n$. The '$X$' term gives a multiple of the inner product

$$(X, \Sigma_1^N H_n.\Delta\tilde{S}_n),$$

which is zero as $X$ is orthogonal to $\tilde{\mathcal{V}}$ and $\Sigma_1^N H_n.\Delta\tilde{S}_n \in \tilde{\mathcal{V}}$. By the Martingale Transform Lemma, $\tilde{S}_n$ is a $P^{**}$-martingale since $H$ (previsible) is arbitrary. Thus $P^{**}$ is a second equivalent martingale measure, different from $P^*$. So incompleteness implies non-uniqueness of equivalent martingale measures, as required. //

*Martingale Representation.* To say that every contingent claim can be replicated means that every $P^*$-martingale (where $P^*$ is the risk-neutral measure, which is unique) can be written, or *represented*, as a martingale transform (of the discounted prices) by the replicating (perfect-hedge) trading strategy $H$. In stochastic-process language, this says that all $P^*$-martingales can be *represented* as martingale transforms of discounted prices. Such Martingale Representation Theorems hold much more generally, and are very important. For the Brownian case, see VI and [RY], Ch. V.
*Note.* In the example of Chapter I, we saw that the simple option there could

be replicated. More generally, in our market set-up, *all* options can be replicated – our market is *complete*. Similarly for the Black-Scholes theory below.

## §4. The Fundamental Theorem of Asset Pricing.

We summarise what we have learned so far. We call a measure $P^*$ under which discounted prices $\tilde{S}_n$ are $P^*$-martingales a *martingale measure*. Such a $P^*$ equivalent to the true probability measure $P$ is called an *equivalent martingale measure*. Then
1 (**No-Arbitrage Theorem**: §2). If the market is *viable* (arbitrage-free), equivalent martingale measures $P^*$ *exist*.
2 (**Completeness Theorem**: §3). If the market is *complete* (all contingent claims can be replicated), equivalent martingale measures are *unique*. Combining:

**Theorem (Fundamental Theorem of Asset Pricing)**. In a complete viable market, there exists a unique equivalent martingale measure $P^*$ (or $Q$).

Let $h$ ($\geq 0$, $\mathcal{F}_N$-measurable) be any contingent claim, $H$ an admissible strategy replicating it:

$$V_N(H) = h.$$

As $\tilde{V}_n$ is the martingale transform of the $P^*$-martingale $\tilde{S}_n$ (by $H_n$), $\tilde{V}_n$ is a $P^*$-martingale. So $V_0(H)(= \tilde{V}_0(H)) = E^*(\tilde{V}_N(H))$. Writing this out in full:

$$V_0(H) = E^*(h/S_N^0).$$

More generally, the same argument gives $\tilde{V}_n(H) = V_n(H)/S_n^0 = E^*[(h/S_N^0)|\mathcal{F}_n]$:

$$V_n(H) = S_n^0 E^*(\frac{h}{S_N^0}|\mathcal{F}_n) \qquad (n = 0, 1, \cdots, N).$$

It is natural to call $V_0(H)$ above the *value* of the contingent claim $h$ at time 0, and $V_n(H)$ above the value of $h$ at time $n$. For, if an investor *sells* the claim $h$ at time $n$ for $V_n(H)$, he can follow strategy $H$ to replicate $h$ at time $N$ and clear the claim. To sell the claim for *any other amount* would provide an arbitrage opportunity (as with the argument for put-call parity). So this value $V_n(H)$ is the *arbitrage price* (or more exactly, *arbitrage-free price* or *no-arbitrage price*); an investor selling for this value is *perfectly hedged*.

We note that, to calculate prices as above, we need to know only
(i) $\Omega$, the set of all possible states,
(ii) the $\sigma$-field $\mathcal{F}$ and the filtration (or information flow) $(\mathcal{F}_n)$,
(iii) the EMM $P^*$ (or $Q$).
We do **NOT** need to know the underlying probability measure $P$ – only its null sets, to know what 'equivalent to $P$' means (actually, in this model, only the empty set is null).

Now option pricing is our central task, and for pricing purposes $P^*$ is vital and $P$ itself irrelevant. We thus may – and shall – focus attention on $P^*$, which is called the *risk-neutral* probability measure. *Risk-neutrality* is the central concept of the subject. The concept of risk-neutrality is due in its modern form to Harrison and Pliska [HP] in 1981 – though the idea can be traced back to actuarial practice much earlier. Harrison and Pliska call $P^*$ the *reference measure*; other names are *risk-adjusted* or *martingale measure*. The term 'risk-neutral' reflects the $P^*$-martingale property of the risky assets, since martingales model fair games.

To summarise, we have the

**Theorem (Risk-Neutral Pricing Formula)**. In a complete viable market, arbitrage-free prices of assets are their discounted expected values under the risk-neutral (equivalent martingale) measure $P^*$ (or $Q$). With payoff $h$,

$$V_n(H) = (1+r)^{-(N-n)} E^*[V_N(H)|\mathcal{F}_n] = (1+r)^{-(N-n)} E^*[h|\mathcal{F}_n].$$

## §5. European Options. The Discrete Black-Scholes Formula.

We consider the simplest case, the Cox-Ross-Rubinstein *binomial model* of 1979; see [CR], [BK].

We take $d = 1$ for simplicity (one risky asset, one riskless asset or bank account); the price vector is $(S_n^0, S_n^1)$, or $((1+r)^n, S_n)$, where

$$S_{n+1} = \begin{cases} S_n(1+a) & \text{with probability } p, \\ S_n(1+b) & \text{with probability } 1-p \end{cases}$$

with $-1 < a < b$, $S_0 > 0$. So writing $N$ for the expiry time,

$$\Omega = \{1+a, 1+b\}^N,$$

each $\omega \in \Omega$ representing the successive values of $T_{n+1} := S_{n+1}/S_n$, $n = 0, 1, \cdots, N-1$. The filtration is $\mathcal{F}_0 = \{\emptyset, \Omega\}$ (trivial $\sigma$-field), $\mathcal{F}_T = \mathcal{F} = 2^\Omega$ (power-set of $\Omega$: class of all subsets of $\Omega$), $\mathcal{F}_n = \sigma(S_1, \cdots, S_n) = \sigma(T_1, \cdots, T_n)$. For $\omega = (\omega_1, \cdots, \omega_N) \in \Omega$, $P(\{\omega_1, \cdots, \omega_N\}) = P(T_1 = \omega_1, \cdots, T_N = \omega_N)$, so knowing the probability measure $P$ (equivalently, knowing $p$) means we know the distribution of $(T_1, \cdots, T_N)$.

For $p^* \in (0,1)$ to be determined, let $P^*$ correspond to $p^*$ as $P$ does to $p$. Then the discounted price $(\tilde{S}_n)$ is a $P^*$-martingale iff

$$E^*[\tilde{S}_{n+1}|\mathcal{F}_n] = \tilde{S}_n, \qquad E^*[(\tilde{S}_{n+1}/\tilde{S}_n)|\mathcal{F}_n] = 1,$$

$$E^*[T_{n+1}|\mathcal{F}_n] = 1 + r \qquad (n = 0, 1, \cdots, N-1),$$

since $S_n = \tilde{S}_n(1+r)^n$, $T_{n+1} = S_{n+1}/S_n = (\tilde{S}_{n+1}/\tilde{S}_n)(1+r)$. But

$$E^*(T_{n+1}|\mathcal{F}_n) = (1+a).p^* + (1+b).(1-p^*)$$

is a weighted average of $1+a$ and $1+b$; this can be $1+r$ iff $r \in [a,b]$. As $P^*$ is to be *equivalent* to $P$ and $P$ has no non-empty null-sets, $r = a, b$ are excluded. Thus by §2:

**Lemma**. The market is viable (arbitrage-free) iff $r \in (a,b)$.

Next, $1+r = (1+a)p^* + (1+b)(1-p^*)$, $r = ap^* + b(1-p^*)$: $r - b = p^*(a-b)$:

**Lemma**. The equivalent martingale measure exists, is unique, and is given by

$$p^* = (b-r)/(b-a).$$

**Corollary**. The market is complete.

Now $S_N = S_n \Pi_{n+1}^N T_i$. By the Fundamental Theorem of Asset Pricing, the price $C_n$ of a call option with strike-price $K$ at time $n$ is

$$\begin{aligned} C_n &= (1+r)^{-(N-n)} E^*[(S_N - K)_+ | \mathcal{F}_n] \\ &= (1+r)^{-(N-n)} E^*[(S_n \Pi_{n+1}^N T_i - K)_+ | \mathcal{F}_n]. \end{aligned}$$

Now the conditioning on $\mathcal{F}_n$ has no effect – on $S_n$ as this is $\mathcal{F}_n$-measurable (known at time $n$), and on the $T_i$ as these are independent of $\mathcal{F}_n$. So

$$\begin{aligned} C_n &= (1+r)^{-(N-n)} E^*[(S_n \Pi_{n+1}^N T_i - K)_+] \\ &= (1+r)^{-(N-n)} \Sigma_{j=0}^{N-n} \binom{N-n}{j} p^{*j} (1-p^*)^{N-n-j} (S_n (1+a)^j (1+b)^{N-n-j} - K)_+; \end{aligned}$$

here $j$, $N-n-j$ are the numbers of times $T_i$ takes the two possible values $1+a, 1+b$. This is the *discrete Black-Scholes formula* of Cox, Ross & Rubinstein (1979) for pricing a European call option in the binomial model. For a European put option, we can either argue similarly or use put-call parity (I.3).

We can find the (perfect-hedge) strategy for replicating this explicitly. Write

$$c(n,x) := \Sigma_{j=0}^{N-n} \binom{N-n}{j} p^{*j}(1-p^*)^{N-n-j}(x(1+a)^j(1+b)^{N-n-j}-K)_+.$$

Then $c(n,x)$ is the undiscounted $P^*$-expectation of the call at time $n$ given that $S_n = x$. This must be the value of the portfolio at time $n$ if the strategy $H = (H_n)$ replicates the claim:

$$H_n^0(1+r)^n + H_n S_n = c(n, S_n)$$

(here by previsibility $H_n^0$ and $H_n$ are both functions of $S_0, \cdots, S_{n-1}$ only). Now $S_n = S_{n-1}T_n = S_{n-1}(1+a)$ or $S_{n-1}(1+b)$, so:

$$\begin{aligned} H_n^0(1+r)^n + H_n S_{n-1}(1+a) &= c(n, S_{n-1}(1+a)) \\ H_n^0(1+r)^n + H_n S_{n-1}(1+b) &= c(n, S_{n-1}(1+b)). \end{aligned}$$

Subtract:

$$H_n S_{n-1}(b-a) = c(n, S_{n-1}(1+b)) - c(n, S_{n-1}(1+a)).$$

So $H_n$ in fact depends only on $S_{n-1}, H_n = H_n(S_{n-1})$ (by previsibility), and

**Proposition**. The perfect hedging strategy $H_n$ replicating the European call option above is given by

$$H_n = H_n(S_{n-1}) = \frac{c(n, S_{n-1}(1+b)) - c(n, S_{n-1}(1+a))}{S_{n-1}(b-a)}.$$

Notice that the numerator is the difference of two values of $c(n,x)$ with the larger value of $x$ in the first term (recall $b > a$). When the payoff function $c(n,x)$ is an increasing function of $x$, as for the European call option considered here, this is non-negative. In this case, the Proposition gives $H_n \geq 0$: the replicating strategy does not involve short-selling. We record this as:

**Corollary**. When the payoff function is a non-decreasing function of the final asset price $S_N$, the perfect-hedging strategy replicating the claim does not involve short-selling of the risky asset.

## §6. Continuous-Time Limit of the Binomial Model.

Suppose now that we wish to price an option in continuous time with initial stock price $S_0$, strike price $K$ and expiry $T$. We can use the work above to give a discrete-time approximation, where $N \to \infty$. Given $R \geq 0$, to be thought of as an instantaneous interest rate in continuous time, define $r$ by

$$r := RT/N : \qquad e^{RT} = \lim_{N\to\infty}(1 + \frac{RT}{N})^N = \lim_{N\to\infty}(1+r)^N.$$

Here $r$, which tends to zero as $N \to \infty$, represents the interest rate in discrete time for the approximating binomial model.
For $\sigma > 0$ fixed ($\sigma^2$ is to be a variance in continuous time, which will correspond to the *volatility* of the stock), define $a, b$ by

$$\log((1+a)/(1+r)) = -\sigma/\sqrt{N}, \qquad \log((1+b)/(1+r)) = \sigma/\sqrt{N}$$

($a, b$ both go to zero as $N \to \infty$). We now have a sequence of binomial models, for each of which we can price options as in §5. We shall show that the pricing formula converges as $N \to \infty$ to a limit (which we shall identify later with the continuous Black-Scholes formula of Ch. VI); see e.g. [BK], 4.6.2.

**Lemma**. Let $(X_j^N)_{j=1}^N$ be iid with mean $\mu_N$ satisfying

$$N\mu_N \to \mu \qquad (N \to \infty)$$

and variance $\sigma^2(1 + o(1))/N$. If $Y_N := \Sigma_1^N X_j^N$, then $Y_N$ converges in distribution to normality:

$$Y_N \to Y = N(\mu, \sigma^2) \qquad (N \to \infty).$$

*Proof.* Use characteristic functions: since $Y_N$ has mean $\mu_N = \mu(1 + o(1))/N$ and variance as given, it also has second moment $\sigma^2(1 + o(1))/N$. So it has characteristic function

$$\phi_N(u) := E\exp\{iuY_N\} = \Pi_1^N E\exp\{iuX_j^N\} = [E\exp\{iuX_1^N\}]^N$$

$$= (1 + \frac{iu\mu}{N} - \frac{1}{2}\frac{\sigma^2 u^2}{N} + o(\frac{1}{N}))^N \to \exp\{iu\mu - \frac{1}{2}\sigma^2 u^2\} \qquad (N \to \infty).$$

This is the characteristic function of the normal law $N(\mu, \sigma^2)$. The result follows, since convergence of characteristic functions implies convergence in distribution by Lévy's continuity theorem for characteristic functions ([W], §18.1). //

We can apply this to pricing the call option above:

$$C_0^{(N)} = (1 + \frac{RT}{N})^{-N} E^*[(S_0 \Pi_1^N T_n - K)_+]$$

$$= E^*[(S_0 \exp\{Y_N\} - (1 + \frac{RT}{N})^{-N} K)_+], \tag{1}$$

where

$$Y_N := \sum_1^N \log(T_n/(1+r)).$$

Since $T_n = T_n^N$ above takes values $1+b, 1+a$, $X_n^N := \log(T_n^N/(1+r))$ takes values $\log((1+b)/(1+r))$, $\log((1+a)/(1+r)) = \pm\sigma/\sqrt{N}$ (so has second moment $\sigma^2/N$). Its mean is

$$\mu_N := \log\Big(\frac{1+b}{1+r}\Big)(1-p^*) + \log\Big(\frac{1+a}{1+r}\Big)p^* = \frac{\sigma}{\sqrt{N}}(1-p^*) - \frac{\sigma}{\sqrt{N}}p^* = (1-2p^*)\sigma/\sqrt{N}$$

(we shall see below that $1 - 2p^* = O(1/\sqrt{N})$, so the Lemma will apply). Now (recall $r = RT/N = O(1/N)$)

$$a = (1+r)e^{-\sigma/\sqrt{N}} - 1, \qquad b = (1+r)e^{\sigma/\sqrt{N}} - 1,$$

so $a, b, r \to 0$ as $N \to \infty$, and

$$1 - 2p^* = 1 - 2\frac{(b-r)}{(b-a)} = 1 - 2\frac{[(1+r)e^{\sigma/\sqrt{N}} - 1 - r]}{[(1+r)(e^{\sigma/\sqrt{N}} - e^{-\sigma/\sqrt{N}})]}$$

$$= 1 - 2\frac{[e^{\sigma/\sqrt{N}} - 1]}{[e^{\sigma/\sqrt{N}} - e^{-\sigma/\sqrt{N}}]}.$$

Now expand the two $[\cdots]$ terms above by Taylor's theorem: they give

$$\frac{\sigma}{\sqrt{N}}(1 + \frac{1}{2}\frac{\sigma}{\sqrt{N}} + \cdots), \qquad \frac{2\sigma}{\sqrt{N}}(1 + \frac{\sigma^2}{6N} + \cdots).$$

So, cancelling $\sigma/\sqrt{N}$,

$$1 - 2p^* = 1 - \frac{2(1 + \frac{1}{2}\frac{\sigma}{\sqrt{N}} + \cdots)}{2(1 + \frac{\sigma^2}{6N} + \cdots)} = -\frac{1}{2}\frac{\sigma}{\sqrt{N}} + O(1/N) :$$

$$N\mu_N = N.\frac{\sigma}{\sqrt{N}}.(-\frac{1}{2}\frac{\sigma}{\sqrt{N}} + O(1/N)) \to \mu := -\frac{1}{2}\sigma^2 \qquad (N \to \infty).$$

We are thus in the situation of the Lemma, with $\mu = -\frac{1}{2}\sigma^2$. In (1), we have $Y_N \to Y$ in distribution and $(1 + \frac{RT}{N})^{-N} \to e^{-RT}$ as $N \to \infty$. This suggests that

$$C_0^{(N)} \to E[(S_0 e^Y - e^{-RT}K)_+],$$

where $E$ is the expectation for the distribution of $Y$, which is $N(-\frac{1}{2}\sigma^2, \sigma^2)$. This can be justified, by standard properties of convergence in distribution (see e.g. [W]). So if $Z := (Y + \frac{1}{2}\sigma^2)/\sigma$, $Z \sim N(0,1)$, $Y = -\frac{1}{2}\sigma^2 + \sigma Z$, and

$$C_0^{(N)} \to \int_{-\infty}^{\infty} [S_0 \exp\{-\frac{1}{2}\sigma^2 + \sigma x\} - e^{-RT}K]_+ \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx \qquad (N \to \infty).$$

To evaluate the integral, note first that $[...] > 0$ where

$$S_0 \exp\{-\frac{1}{2}\sigma^2 + \sigma x\} > e^{-RT}K, \qquad -\frac{1}{2}\sigma^2 + \sigma x > \log(K/S_0) - RT :$$

$$x > [\log(K/S_0) + \frac{1}{2}\sigma^2 - RT]/\sigma = c, \quad \text{say.}$$

So writing $\Phi(x)$ for the standard normal distribution function,

$$C_0 = S_0 \int_c^{\infty} e^{-\frac{1}{2}\sigma^2}. \exp\{-\frac{1}{2}x^2 + \sigma x\} dx/\sqrt{2\pi} - Ke^{-RT}[1 - \Phi(c)].$$

The remaining integral is

$$\int_c^{\infty} \exp\{-\frac{1}{2}(x - \sigma)^2\} dx/\sqrt{2\pi} = \int_{c-\sigma}^{\infty} \exp\{-\frac{1}{2}u^2\} du/\sqrt{2\pi} = 1 - \Phi(c - \sigma).$$

So the option price is given as a function of the initial price $S_0$, strike price $K$, expiry $T$, interest rate $R$ and variance $\sigma^2$ by

$$C_0 = S_0[1 - \Phi(c - \sigma)] - Ke^{-RT}[1 - \Phi(c)], \qquad c = [\log(K/S_0) + \frac{1}{2}\sigma^2 - RT]/\sigma.$$

To compare with our later work, it is convenient now to replace $\sigma^2$ by $\sigma^2 T$; thus $\sigma^2$ is now the variance per unit time. Its square root, $\sigma$, is called the *volatility* of the stock. Then $c - \sigma$, $c$ above become $c_\pm$, where

$$c_\pm := [\log(K/S_0) - (R \pm \frac{1}{2}\sigma^2)T]/\sigma\sqrt{T}.$$

The result extends immediately to give the price of the option at time $t \in (0, T)$, by replacing $T$ by $T - t$, $S_0$ by $S_t$.

We re-write the formula in more customary notation. First, write $r$ in place of $R$ for the interest rate. Next, using the symmetry of the normal distribution, $1 - \Phi(c_\pm) = \Phi(-c_\pm) = \Phi(d_\pm)$, say, where

$$d_\pm := -c_\pm = [\log(S/K) + (r \pm \frac{1}{2}\sigma^2)(T - t)]/\sigma\sqrt{T - t} :$$

the price of the European call option is

$$c_t = S_t \Phi(d_+) - e^{-r(T-t)} K \Phi(d_-).$$

This is the famous *continuous Black-Scholes formula*. We shall return to it in Chapter VI, where we re-derive it by continuous-time methods (Brownian motion and Itô calculus).

*Note.* 1. The same argument (or put-call parity) gives the value of the European put option as $p_t = K e^{-r(T-t)} \Phi(-d_-) - S_t \Phi(-d_+)$.

2. The proof above starts from a binomial distribution and ends with a normal distribution. The binomial distribution is that of a sum of independent Bernoulli random variables. That sums (equivalently, averages) of independent random variables with finite means and variances gives a normal limit is the content of the Central Limit Theorem or CLT (the *Law of Errors*, as physicists would say). The particular form of the CLT used here - normal approximation to the binomial - is the *de Moivre-Laplace limit theorem*.

The picture for this is familiar. The Binomial distribution $B(n, p)$ has a histogram with $n + 1$ bars, whose heights peak at the mode and decrease to either side. For large $n$, one can draw a smooth curve through the histogram. The curve looks like a normal density curve (with the appropriate location and scale, i.e. mean and variance). The result proved above, and the classical de Moivre-Laplace limit theorem, say that this is exactly right.

3. The Cox-Ross-Rubinstein binomial model above goes over in the passage to the limit to the geometric Brownian motion model of VI.1. We will later

re-derive the continuous Black-Scholes formula in Ch. VI, using continuous-time methods (Itô calculus), rather than using the method above of deriving the discrete Black-Scholes formula and going to the limit on the *formula*, rather than the *model*.
4. For similar derivations of the discrete Black-Scholes formula and the passage to the limit to the continuous Black-Scholes formula, see e.g. [CR], §5.6.
5. One of the most striking features of the Black-Scholes formula is that it does **not** involve the mean rate of return $\mu$ of the stock - only the riskless interest-rate $r$ and the volatility of the stock $\sigma$. Mathematically, this reflects the fact that the change of measure involved in the passage to the risk-neutral measure involves a change of drift. This has the effect of eliminating the $\mu$ term; see Ch. VI.
6. The Black-Scholes formula involves the parameter $\sigma$ (where $\sigma^2$ is the variance of the stock per unit time), called the *volatility* of the stock. In financial terms, this represents how sensitive the stock-price is to new information - how 'volatile' the market's assessment of the stock is. This volatility parameter is very important, *but* we do not know it; instead, we have to *estimate* the volatility for ourselves. There are two approaches:
(a) *historic volatility*: here we use Time Series methods to estimate $\sigma$ from past price data. Clearly the more variability we observe in runs of past prices, the more volatile the stock price is, and given enough data we can estimate $\sigma$ in this way.
(b) *implied volatility*: match observed option prices to theoretical option prices. For, the price we see options traded at tells us what the *market* thinks the volatility is (estimating volatility this way works because the dependence is monotone; see later).

If the Black-Scholes model were perfect, these two estimates would agree (to within sampling error). But discrepancies can be observed, which shows the imperfections of our model.

Volatility estimation is a major topic, both theoretically and in practice. We return to this in IV.7.3-4 below and VI.7.5-8. But looking ahead:
(i) trading is itself one of the major causes of volatility;
(ii) options like volatility [i.e., option prices go up with volatility].
Recalling Ch. I, this shows that volatility is a 'bad thing' from the point of view of the real economy (uncertainty about, e.g., future material costs is nothing but a nuisance to manufacturers), but a 'good thing' for financial markets (trading increases volatility, which increases option prices, which generates more trade ...) – at the cost of increased instability.