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§7. More on European Options

1. *Bounds.* We use the notation above. We also write c, p for the values of European calls and puts, C, P for the values of the American counterparts.

Obvious upper bounds are $c \leq S, C \leq S$, where S is the stock price (we can buy for S on the market without worrying about options, so would not pay more than this for the right to buy). For puts, one has correspondingly the obvious upper bounds $p \leq K, P \leq K$, where K is the strike price: one would not pay more than K for the right to sell at price K, as one would not spend more than one's maximum return. A lower bound is given by

Proposition 1. A lower bound on the price c_0 of a European call option at time 0 is

$$c_0 \geq \max(S_0 - Ke^{-rT}, 0).$$

Proof. $c_0 \geq 0$ as options confer rights, not obligations, so it remains to show $c_0 \geq S_0 - Ke^{-rT}$.

Consider the following two portfolios:
I: one European call option plus Ke^{-rT} in cash,
II: one share.
The cash in portfolio I, invested from time 0 to time T, is worth K at time T.
(a) If $S_T \geq K$, we exercise the European call option at time T: we buy the stock with the cash K, and the portfolio is worth S_T.
(b) If $S_T \leq K$, the option is worthless, and the portfolio is worth K.
So at time T, portfolio I is worth $\max(S_T, K)$. Portfolio II is worth S_T. So at time T, I is worth at least as much as II [and sometimes more].

This suggests that the same is true at time 0:

$$c_0 + Ke^{-rT} \geq S_0.$$

If not, and

$$c_0 + Ke^{-rT} < S_0,$$

we show that there is an arbitrage opportunity, as follows: the stock S_0 is *too dear*, so we *sell the stock short*. From the proceeds S_0 of the short sale, we *lock in* $S_0 - [c_0 + Ke^{-rT}] > 0$ in cash (it will be seen to be riskless profit).

Use the remaining $c_0 + Ke^{-rT}$ to *buy* a call option (for c_0), and retain the remaining Ke^{-rT} in cash. At time T, we have the call, and the cash has grown to K, so we have Portfolio I, worth $\max(S_T, K)$. Close out the short position [= deliver the stock, i.e. its value S_T]. We have a cash balance of $\max(0, K - S_T)$ [always ≥ 0, > 0 when $S_T < K$ and the stock is low], in addition to our earlier profit.

This contradicts our standing assumption that the market is arbitrage-free, and so completes the proof. //

The corresponding lower bound for European puts can be found from this and put-call parity (I.7). But we include a proof, for more practice in arbitrage arguments.

Proposition 2. The value p_0 of a European put option at time 0 satisfies

$$p_0 \geq \max(Ke^{-rT} - S_0, 0).$$

Proof. As above, consider the following two portfolios:
I': One European put option and one share,
II': Cash Ke^{-rT}.
(a) If $S_T < K$, the put option in I' is exercised at time T, and the portfolio is worth K.
(b) If $S_T > K$, the put option is worthless, and the portfolio is worth S_T.
So the value of I' at T is $\max(S_T, K)$. The value of II' at T is K. So I' is always worth at least as much as II' at T [and sometimes more]. This suggests that the same is true at time 0:

$$p_0 + S_0 \geq Ke^{-rT}.$$

If not, and $Ke^{-rT} > p_0 + S_0$: buy Portfolio I' - one share and one put - and borrow Ke^{-rT}. Lock in $Ke^{-rT} - (p_0 + S_0) > 0$ [it will prove to be riskless profit]. At time T, we have portfolio I', worth $\max(S_T, K)$, and our debt has grown to K. Clear the debt. We have cash balance $\max(0, S_T - K)$ [always ≥ 0, > 0 if $S_T > K$ and the stock is high], in addition to our earlier profit. This arbitrage opportunity contradicts the arbitrage-free assumption on the market, and we conclude as before. //

2. *Dependence of the Black-Scholes price on the parameters.*

Recall the Black-Scholes formulae for the values c_t, p_t for the European call and put:

$$c_t = S_t \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-), \qquad p_t = K e^{-r(T-t)} \Phi(-d_-) - S_t \Phi(-d_+),$$

where

$$d_\pm := [\log(S_t/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{(T-t)}.$$

1. S. As the stock price S increases, the call option becomes more and more likely to be exercised. In the limit for large S, $d_\pm \to \infty$, $\Phi(d_\pm) \to 1$, so $c_t \to S_t - Ke^{-r(T-t)}$. This limit has a natural economic interpretation: it is the value of a *forward contract* with *delivery price* K (see e.g. Hull [H1] Ch. 3, [H2] Ch. 3).
2. σ. When the volatility $\sigma \to 0$, the stock becomes riskless, and behaves like money in the bank. Again, $d_\pm \to \infty$, the Black-Scholes price has the limit above, and one has the correct economic interpretation.

3. *Volatility.*

As in IV.6.6 Week 6, the volatility σ can be estimated in two ways:
a. Directly from the movement of a stock price in time [as the mathematics here is continuous time, we defer it to Ch. VI], giving what is called the *historic volatility.*
b. From the observed market prices of options: if we know everything in the Black-Scholes formula (including the price at which the option is traded) except the volatility σ, we can solve for σ. This is called *implied volatility.* Since σ appears inside the argument of the normal distribution function Φ as well as outside it, this is a transcendental equation for σ and has to be solved numerically by iteration (Newton-Raphson method). We quote that the Black-Scholes price is a monotone (increasing) function of the volatility (more volatility doesn't make us 'more likely to win', but when we do win, we 'win bigger'), so there is a unique root of the equation.

In practice, one encounters discrepancies between historic and implied volatility, which show limitations to the accuracy of the Black-Scholes model. It is, however, the standard 'benchmark model', and is valuable as a first approximation.

The classical view of volatility is that it is caused by future uncertainty, and shows the market's reaction to the stream of new information. However, studies taking into account periods when the markets are open and closed [there are only about 250 trading days in the year] have shown that the volatility is less when markets are closed than when they are open. This

suggests that *trading itself is one of the main causes of volatility.*
Note. This observation has deep implications for the macro-prudential and regulatory issues discussed in Ch. 1. The real economy cannot afford too much volatility. Volatility is (at least partly) caused by trading. Conclusion: there is too much trading. Policy question: how can we reduce the volume of trading (much of it speculative, designed to enrich traders, and not serving a more widely useful economic purpose)? One answer is the so-called *Tobin tax* (also known as the "Robin Hood tax") (James Tobin (1918-2002), American economist; Nobel Prize for Economics, 1981). This would levy a small charge (e.g. 0.01%) on *all* financial transactions. This would both provide a major and useful source of tax revenue, and – more importantly – would discourage a lot of speculative trading, thereby (shrinking the size of the financial services industry, but) diminishing volatility, to the benefit of the general economy (Problems 7 again).

4. *The Greeks.*

These are the partial derivatives of the option price with respect to the input parameters. They have the interpretation of *sensitivities.*
(i) For a call, say, $\partial c/\partial S$ is called the *delta*, Δ. Adjusting our holdings of stock to eliminate our portfolio's dependence on S (to first order) is called *delta-hedging.*
(ii) Second-order effects involve *gamma* $:= \partial(\Delta)/\partial S$.
(iii) Time-dependence is given by *Theta* is $\partial c/\partial t$.
(iv) Volatility dependence is given by *vega* $:= \partial c/\partial \sigma$.

From the Black-Scholes formula (which gives the price explicitly as a function of σ), one can check by calculus (Problems 7) that

$$\partial c/\partial \sigma > 0,$$

and similarly for puts (or, use the result for calls and put-call parity). To summarise: *options like volatility.* This is entirely in line with one's intuition. The more uncertain things are (the higher the volatility), the more valuable protection against adversity becomes (the higher the option price).
(v) *rho* is $\partial c/\partial r$, the sensitivity to interest rates.

§8. American Options.

We now consider an American call option (value C), in the simplest case of a stock paying no dividends. The following result goes back (at least) to

R. C. MERTON in 1973.

Theorem. It is never optimal to exercise an American call option early. That is, the American call option is equivalent to the European call, so has the same value:

$$C = c.$$

First Proof. Consider the following two portfolios:
I: one American call option plus cash Ke^{-rT},
II: one share.
The value of the cash in I is K at time T, $Ke^{-r(T-t)}$ at time t. If the call option is exercised early at $t < T$, the value of Portfolio I is then $S_t - K$ from the call, $Ke^{-r(T-t)}$ from the cash, total

$$S_t - K + Ke^{-r(T-t)}.$$

Since $r > 0$ and $t < T$, this is $< S_t$, the value of Portfolio II at t. So Portfolio I is *always* worth less than Portfolio II if exercised *early*.

If however the option is exercised instead at expiry, T, the American call option is then the same as a European call option. We are then in the situation of Proposition 1 of §7: at time T, Portfolio I is worth $\max(S_T, K)$ and Portfolio II is worth S_T. So:

$$\begin{array}{lll} \text{before} & T, & I < II, \\ \text{at} & T, & I \geq II \text{ always, and } > \text{ sometimes.} \end{array}$$

This direct comparison with the underlying [the share in Portfolio II] shows that early exercise is never optimal. Since an American option at expiry is the same as a European one, this completes the proof. //
Second Proof. Since American options confer all the rights of European options, and more, they must be worth at least as much: $C \geq c$.

Now by Proposition 1 of §6, $c_0 \geq S_0 - Ke^{-rT}$. This and $C_0 \geq c_0$ give $C_0 \geq S_0 - Ke^{-rT}$. Using $t < T$ as initial time instead of 0: $C_t \geq S_t - Ke^{-r(T-t)}$. Now $r > 0$ and $t < T$, so $Ke^{-r(T-t)} < K$. This gives

$$C_t > S_t - K.$$

Now if it were optimal to exercise early at $t < T$, the value of the American call (the amount it would yield) would be $S_t - K$. So we would have $C_t = S_t - K$. This would contradict the inequality above, so early exercise can never be optimal. //

Economic Interpretation. There are two reasons why an American call should not be exercised early:
1. *Insurance.* Consider an investor choosing to hold a call option instead of the underlying stock. He does not care if the share price falls below the strike price (as he can then just discard his option) – but if he held the stock, he would. Thus the option insures the investor against such a fall in stock price, and if he exercises early, he loses this insurance.
2. *Interest on the strike price.* When the holder exercises the option, he buys the stock and pays the strike price, K. Early exercise at $t < T$ deprives the holder of the interest on K between times t and T: the later he pays out K, the better.

American Puts.

Recall the put-call parity of Ch. I (valid only for European options):

$$c - p = S - Ke^{-rT}.$$

A partial analogue for American options is given by the inequalities below:

$$S - K < C - P < S - Ke^{-rT}.$$

For proof (similar to those above) and background, see e.g. Ch. 8 (p. 216) of [H1].

We now consider how to evaluate an American put option, European and American call options having been treated already. First, we will need to work in discrete time. We do this by dividing the time-interval $[0, T]$ into N equal subintervals of length Δt say. Next, we take the values of the underlying stock to be discrete: we use the binomial model of §5, with a slight change of notation: we write u, d ('up', 'down') for $(1 + b), (1 + a)$: thus stock with initial value S is worth $Su^i d^j$ after i steps up and j steps down. Consequently, after N steps, there are $N + 1$ possible prices, $Su^i d^{N-i}$ $(i = 0, \cdots, N)$. It is convenient to display the possible paths followed by the stock price as a binomial tree [see diagram], with time going left to right and two paths, up and down, leaving each node in the tree, until we reach the $N + 1$ terminal nodes at expiry. There are 2^N possible paths through the tree. It is common to take N of the order of 30, for two reasons:
(i) typical lengths of time to expiry of options are measured in months (9

months, say); this gives a time-step around the corresponding number of days,
(ii) 2^{30} paths is about the order of magnitude that can be comfortably handled by computers (recall that $2^{10} = 1,024$, so 2^{30} is somewhat over a billion).

We now return to our treatment of the binomial model in §§5,6, with a slight change of notation. Recall that in §5 (discrete time) we used $1 + r$ for the discount factor. It is convenient to call this $1 + \rho$ instead, freeing r for its usual use as the short rate of interest in continuous time. Thus $1 + \rho = e^{r\Delta t}$, and the risk-neutrality condition $p^* = (b - r)/(b - a)$ of §5 becomes

$$p^* = (u - e^{r\Delta t})/(u - d).$$

Now recall (§7) $(1+a)/(1+r) = \exp(-\sigma/\sqrt{N})$, $(1+b)/(1+r) = \exp(\sigma/\sqrt{N})$. We replaced σ^2 by $\sigma^2 T$ (to make σ the volatility per unit time), and $T = N.\Delta t$, so $\sigma/\sqrt{N}$ becomes $\sigma\sqrt{T}/\sqrt{N} = \sigma\sqrt{\Delta t}$. So now

$$u/e^{r\Delta t} = e^{\sigma/\sqrt{\Delta t}}, \qquad d/e^{r\Delta t} = e^{-\sigma\sqrt{\Delta t}}.$$

Thus $ud = e^{2r\Delta t}$. Since $\sqrt{\Delta t}$ is small, its square Δt is a second-order term; to first order, we thus have $ud = 1$, which simplifies filling in the terminal values in the binary tree.

With an eye on this simplification, we begin again: define our up and down factors u, d so that

$$ud = 1;$$

define the risk-neutral probability p^* so as to have

$$p^* = (u - e^{r\Delta t})/(u - d)$$

(so as to get the mean return from the risky stock the same as that from the riskless bank account), and the volatility σ so as to have the variance of the value S' of the stock after one time-step when it is worth S initially as $S^2\sigma^2\Delta t$:

$$S^2\sigma^2\Delta t = p^* S^2 u^2 + (1 - p^*)S^2 d^2 - S^2[p^* u + (1 - p^*)d]^2$$

(using $var S' = E(S'^2) - [ES']^2$). Then to first order in $\sqrt{\Delta t}$ (which is all the accuracy we shall need), one can check that we have

$$u = \exp(\sigma\sqrt{\Delta t}), \qquad d = \exp(-\sigma\sqrt{\Delta t})$$

as before.

We can now calculate both the value of an American put option and the optimal exercise strategy by working backwards through the tree (this method of backward recursion in time is a form of the Dynamic Programming [DP] technique, due to Richard Bellman, which is important in many areas of optimization and Operational Research).

1. Draw a binary tree showing the initial stock value and having the right number, N, of time-intervals.

2. Fill in the stock prices: after one time interval, these are Su (upper) and Sd (lower); after two time-intervals, Su^2, S and $Sd^2 = S/u^2$; after i time-intervals, these are $Su^j d^{i-j} = Su^{2j-i}$ at the node with j 'up' steps and $i-j$ 'down' steps (the '(i,j)' node).

3. Using the strike price K and the prices at the *terminal nodes*, fill in the payoffs ($f_{N,j} = \max[K - Su^j d^{N-j}, 0]$) from the option at the terminal nodes (where, at expiry, the values of the European and American options coincide) underneath the terminal prices.

4. Work back down the tree one time-step. Fill in the 'European' value at the penultimate nodes as the discounted values of the upper and lower right (terminal node) values, under the risk-neutral measure - 'p^* times lower right plus $1-p^*$ times upper right' [notation of IV.6 Week 6]. Fill in the 'intrinsic' (or early-exercise) value - the value if the option is exercised. Fill in the American put value as the higher of these.

5. Treat these values as 'terminal node values', and fill in the values one time-step earlier by repeating Step 4 for this 'reduced tree'.

6. Iterate. The value of the American put at time 0 is the value at the root - the last node to be filled in. The 'early-exercise region' is the set of nodes where the early-exercise value is the higher; the remaining set of nodes is the 'continuation region'.

Note. The above procedure is simple to describe and understand, and simple to programme. It is laborious to implement numerically by hand, on examples big enough to be non-trivial. Numerical examples are worked through in detail in [H1], 359-360 and [CR], 241-242.

Mathematically, the task remains of describing the *continuation region* - the part of the tree where early exercise is not optimal. This is a classical *optimal stopping problem.* It would take us too far afield to pursue this; for a fairly thorough (but quite difficult) treatment, see [SKKM], I. We will, however, connect the work above with that of Ch. III on the Snell envelope. Consider the pricing of an American put, strike price K, expiry N, in discrete time, with discount factor $1+r$ per unit time as earlier. Let $Z = (Z_n)_{n=0}^N$ be the payoff on exercising at time n. We want to price Z_n, by U_n say (to conform to our earlier notation), so as to avoid arbitrage; again, we work backwards in time. The recursive step is

$$U_{n-1} = \max(Z_{n-1}, \frac{1}{1+r} E^*[U_n|\mathcal{F}_{n-1}]),$$

the first alternative on the right corresponding to early exercise, the second to the discounted expectation under P^*, as usual. Let $\tilde{U}_n = U_n/(1+r)^n$ be the discounted price of the American option: then

$$\tilde{U}_{n-1} = \max(\tilde{Z}_{n-1}, E^*[\tilde{U}_n|\mathcal{F}_{n-1}]).$$

Thus $(\tilde{U}_n)$ is the *Snell envelope* (III.7) of the discounted payoff process $(\tilde{Z}_n)$. It is thus:
(i) a P^*-supermartingale,
(ii) the smallest supermartingale dominating $(\tilde{Z}_n)$,
(iii) the solution of the optimal stopping problem for $\tilde{Z}$.

We conclude by showing the equivalence of American and European calls without using arbitrage arguments.

Theorem. Let $(Z_n)_0^N$ be an adapted sequence, $h := Z_N$; write C_n, c_n for the values at time n of the American and European options generated by the payoff function h. Then
(i) $C_n \geq c_n$,
(ii) If $c_n \geq Z_n$, then $C_n = c_n$.

Proof. (i)

$$\begin{aligned} \tilde{C}_n &\geq E^*[\tilde{C}_N|\mathcal{F}_n] \qquad ((\tilde{C}_k) \text{ a } P^*\text{-supermartingale}) \\ &= E^*[\tilde{c}_N|\mathcal{F}_n] \qquad (C_N = c_N) \\ &= \tilde{c}_n \qquad ((\tilde{c}_k) \text{ a } P^*\text{-martingale}). \end{aligned}$$

(ii) $(\tilde{c}_n)$ is a P^*-martingale, so in particular a P^*-supermartingale. Being the Snell envelope of (Z_n), $(\tilde{C}_n)$ is the *least* P^*-supermartingale dominating (Z_n). So if $\tilde{c}_n \geq Z_n$ as in the condition of the theorem, $\tilde{c}_n \geq \tilde{C}_n$, so $\tilde{c}_n = \tilde{C}_n$. //

Corollary. In the Black-Scholes model with one risky asset, the American call option is equivalent to its European counterpart.

Proof. Here $Z_n = (S_n - K)_+$. Discounting,

$$\begin{aligned} \tilde{c}_n &= (1+r)^{-N} E^*[(S_N - K)_+ | \mathcal{F}_n] \\ &\geq E^*[\tilde{S}_N - K(1+r)^{-N} | \mathcal{F}_n] \qquad (x_+ \geq x, \text{ so } Ex_+ \geq Ex) \\ &= \tilde{S}_n - K(1+r)^{-N} \qquad (\tilde{S}_n \text{ a } P^*\text{-martingale}). \end{aligned}$$

Without the discounting, this says

$$c_n \geq S_n - K(1+r)^{-(N-n)}.$$

This gives $c_n \geq S_n - K$; also $c_n \geq 0$; so $c_n \geq (S_n - K)_+ = Z_n$, and the result follows from the Theorem. //

For a survey of American options, see

[M] MYNENI, R. (1992): The pricing of the American option. *Ann. Appl. Probab.* **2**, 1-23.

§9. American options – infinite horizon.

We sketch here the theory of the American option (one can exercise at any time), over an infinite time-horizon (motivated by Ch. VI below on *real options*). We deal with a *put* option (see Ch. VI below for the call option) – giving the right to sell at the strike price K, at any time τ of our choosing. This τ has to be a *stopping time*: we have to take the decision whether or not to stop at τ based on information already available (that is, contained in $\mathcal{F}_\tau$ – no access to the future, no insider trading). As above, we pass to the risk-neutral measure.

The theory is easier in continuous time (Ch. VI), but as it links so well with §8 above we include it here, referring forward to Ch. VI for *stochastic differential equations (SDEs)* and *geometric Brownian motion (GBM)*. Under the risk-neutral measure, the SDE for GBM becomes

$$dX_t = rX_t dt + \sigma X_t dB_t. \qquad (GBM_r)$$

To evaluate the option, we have to solve the *optimal stopping problem*

$$V(x) := \sup_\tau E_x[e^{-r\tau}(K - X_\tau)^+]$$

where the supremum is taken over all stopping times τ and $X_0 = x$ under P_x.

The process X satisfying (GBM_r) is specified by a second-order linear differential operator, called its (infinitesimal) *generator*,

$$L_X := rxD + \frac{1}{2}\sigma^2 x^2 D^2, \qquad D := \partial/\partial x.$$

Now the closer X gets to 0, the less likely we are to gain by continuing. This suggests that our best strategy is to stop when X gets too small: to stop at $\tau = \tau_b$, where

$$\tau_b := \inf\{t \geq 0 : X_t \leq b\},$$

for some $b \in (0, K)$. This gives the following *free boundary problem* for the *unknown value function* $V(x)$ and the *unknown point* b:

$$L_X V = rV \qquad \text{for } x > b; \tag{i}$$

$$V(x) = (K - x)^+ \qquad \text{for } x = b; \tag{ii}$$

$$V'(x) = -1 \qquad \text{for } x = b \text{ (smooth fit)}; \tag{iii}$$

$$V(x) > (K - x)^+ \qquad \text{for } x > b; \tag{iv}$$

$$V(x) = (K - x)^+ \qquad \text{for } 0 < x < b. \tag{v}$$

Writing $d := \sigma^2/2$ ('d for diffusion'), (i) is

$$dx^2 V'' + rxV' - rV = 0. \tag{i^*}$$

Trial solution:

$$V(x) = x^p.$$

Substituting gives a quadratic for p:

$$p^2 - (1 - \frac{r}{d})p - \frac{r}{d} = 0.$$

One root is $p = 1$; the other is $p = -r/d$. So the general solution (GS) to the DE (i^*) is

$$V(x) = C_1 x + C_2 x^{-r/d},$$

for some constants C_1 and C_2. But $V(x) \leq K$ for all $x \geq 0$ (an option giving the right to sell at price K cannot be worth more than K!), so $C_1 = 0$. This gives

$$C_2 = \frac{d}{r}\Big(\frac{K}{1 + d/r}\Big)^{1+r/d}, \qquad b = \frac{K}{1 + d/r}.$$

So

$$\begin{aligned} V(x) &= \frac{d}{r}\Big(\frac{K}{1+d/r}\Big)^{1+r/d} x^{-r/d} \qquad \text{if } x \in [b,\infty) \\ &= K - x \qquad \text{if } x \in (0,b]. \end{aligned}$$

This is in fact the full and correct solution to the problem. For details, see [P&S], §25.1.

The 'smooth fit' in (iii) is characteristic of free boundary problems. For a heuristic analogy: imagine trying to determine the shape of a rope, tied to the ground on one side of a convex body, stretched over the body, then pulled tight and tied to the ground on the other side. We can see on physical grounds that the rope will be:
straight to the left of the convex body;
continuously in contact with the body for a while, then
straight to the right of the body, and
there should be no kink in the rope at the points where it makes and then leaves contact with the body.
This 'no kink' condition corresponds to 'smooth fit' in (iii).

Discrete v. continuous models in mathematical finance.

In Ch. IV we have studied discrete models, obtaining (discrete versions of) the No-Arbitrage and Completeness Theorems and the FTAP, and proving everything. In Ch. VI we study continuous models, obtaining continuous versions of these results and proving as much as possible – but not everything. One mathematical difference between IV and VI is in Measure Theory (which we could do without in IV, but not in VI). Another is seen in the Separating Hyperplane Theorem (SHT), used to prove the NA Theorem in IV.2. In IV, spaces are finite-dimensional, and the SHT is just Euclidean geometry. (We have a cone, with axis L say. The required hyperplane is that through the origin perpendicular to L.) In VI, spaces are infinite-dimensional, and now the SHT needs the Hahn-Banach Theorem, the cornerstone of Functional Analysis. In turn, the Hahn-Banach Theorem needs the Axiom of Choice (AC), or some variant of it such as Zorn's Lemma.

Note carefully the results needing SHT or AC. These tend to be the hard ones, and the crucial ones.