

MATH482 EXAMINATION, 2014

Six questions, do four; 25 marks each

Q1. Discuss briefly:

- (i) arbitrage;
- (ii) completeness of markets;
- (iii) equivalent martingale measures;
- (iv) risk-neutral valuation.

Q2. State and prove the put-call parity relationship between prices of call and put options.

Q3. Analyze the ‘doubling strategy’: when betting on tossing a fair coin, respond to losing by doubling the stakes.

Show that this leads to an eventual certain gain. Explain why this does not work in practice as a means of making money.

Q4. (i) Given a random variable Y with $E[|Y|] < \infty$ and a σ -field \mathcal{C} , define the conditional expectation $E[Y|\mathcal{C}]$ of Y with respect to \mathcal{C} .

(ii) What happens if \mathcal{C} is the trivial σ -field? What happens if \mathcal{C} is the whole σ -field?

(iii) Show that if \mathcal{B}, \mathcal{C} are σ -fields with $\mathcal{B} \subset \mathcal{C}$, then applying both conditional expectations is the same as applying just one of them, either way round. Which one?

(iv) Show that the expectation of a conditional expectation is just the expectation.

(v) Explain what is meant by saying that a conditional expectation is a projection.

Q5. Given the Black-Scholes formula for a call price,

$$C_t := S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \quad (BS)$$

where

$$\phi(x) := e^{-\frac{1}{2}x^2} / \sqrt{2\pi}, \quad \Phi(x) := \int_{-\infty}^x \phi(u) du$$

are the standard normal density and distribution functions, $\tau := T - t$ the

time to expiry,

$$d_1 := \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 := \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau},$$

show that

(i) $\phi(d_2) = \phi(d_1) \cdot e^{d_1\sigma\sqrt{\tau}} \cdot e^{-\frac{1}{2}\sigma^2\tau}$;

(ii) $e^{d_1\sigma\sqrt{\tau}} = (S/K) \cdot e^{r\tau} \cdot e^{\frac{1}{2}\sigma^2\tau}$;

(iii) $\phi(d_2) = \phi(d_1) \cdot (S/K) \cdot e^{r\tau}$: $Ke^{-r\tau}\phi(d_2) = S\phi(d_1)$.

Hence show that vega (the partial derivative of the option price with respect to the volatility σ) is positive. Obtain the same result for put option prices. Give the financial interpretation of these results.

Q6. (i) In the Black-Scholes model with riskless interest rate r and one risky stock with mean return rate μ and volatility σ , define the Sharpe ratio λ , and discuss how a fund manager would decide the proportion of his funds to be invested in the risky stock.

(ii) Describe briefly, without proofs, how to pass from the dynamics of the risky stock to the Black-Scholes formula for the price of options on it.

(iii) Why does the Black-Scholes formula not involve μ ?

(iv) The Black-Scholes price does depend on σ , but we do not know it: how does one estimate σ ?

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