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PROBLEMS 4. 17.2.2014

The Bivariate Normal Distribution. Define

$$f(x,y) = c\exp\{-\frac{1}{2}Q(x,y)\},$$

where c is a constant, Q a positive definite quadratic form in x and y. Specifically:

$$c = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}, \qquad Q = \frac{1}{1-\rho^2}\Big[\Big(\frac{x-\mu_1}{\sigma_1}\Big)^2 - 2\rho\Big(\frac{x-\mu_1}{\sigma_1}\Big)\Big(\frac{y-\mu_2}{\sigma_2}\Big) + \Big(\frac{y-\mu_2}{\sigma_2}\Big)^2\Big].$$

Here $\sigma_i > 0$, μ_i are real, $-1 < \rho < 1$. Show that:

Q1. f is a probability density – that is, that f is non-negative and integrates to 1.

Q2. If f is the density of a random 2-vector (X,Y), X and Y are normal, with distributions $N(\mu_1, \sigma_1^2)$, $N(\mu_2, \sigma_2^2)$.

Q3. X, Y have means μ_1, μ_2 and variances σ_1^2, σ_2^2.

Q4. The conditional distribution of y given $X = x$ is $N(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1), \quad \sigma_2^2(1-\rho^2))$.

Q5. The conditional mean $E(Y|X=x)$ is *linear* in x:

$$E(Y|X=x) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1).$$

NHB