

PROBLEMS 9. 24.3.2014

Q1 *Brownian covariance.* The *covariance* of random variables X, Y is

$$\text{cov}(X, Y) := E[(X - E[X])(Y - E[Y])].$$

Show that for $B = (B_t)$ Brownian motion (BM), its covariance is

$$\text{cov}(B_s, B_t) = \min(s, t).$$

We quote that for a Gaussian process (one all of whose finite-dimensional distributions are Gaussian, such as BM), the process is characterised by its mean function and covariance function (so mean 0 and covariance $\min(s, t)$ characterise BM).

Q2 *Brownian scaling.* With $c > 0$ and B Brownian motion, show that B_c , where

$$B_c(t) := B(c^2 t)/c,$$

has the same covariance function $\min(s, t)$ as Brownian motion B . Deduce that (as B_c is also continuous and Gaussian) that B_c is Brownian motion. It is formed from B by *Brownian scaling*.

Deduce that B is *self-similar*: it reproduces itself if time and space are both scaled as above. We call such a self-similar process a *fractal*.

If Z is the zero-set of B and Z_c that of B_c , deduce that Z also is a fractal.

Q3 *Time inversion.* For B BM, and

$$X_t := tB(1/t) \quad (t \neq 0),$$

$X = (X_t)$ is also BM.

Deduce or prove otherwise that for B BM

$$B(t)/t \rightarrow 0 \quad (t \rightarrow \infty).$$

Q4. By writing

$$\begin{aligned}
\int_0^t B(u)dB(u) &= \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} B(kt/n)(B((k+1)t/n) - B(kt/n)) \\
&= \sum \frac{1}{2}(B((k+1)t/n) + B(kt/n)).(B((k+1)t/n) - B(kt/n)) \\
&\quad - \sum \frac{1}{2}(B((k+1)t/n) - B(kt/n)).(B((k+1)t/n) - B(kt/n)),
\end{aligned}$$

or otherwise, show that

$$\int_0^t B(u)dB(u) = \frac{1}{2}B(t)^2 - \frac{1}{2}t.$$

Comment on the difference between this Itô calculus result and ordinary (Newton-Leibniz) calculus.

NHB