livsoln4.tex

SOLUTIONS 4. 24.2.2014

Q1. Since f is clearly non-negative, to show that f is a (probability density) function (in two dimensions), it suffices to show that f integrates to 1:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1, \qquad \text{or} \qquad \int\int f = 1.$$

Write

$$f_1(x) := \int_{-\infty}^{\infty} f(x,y)dy, \qquad f_2(y) := \int_{-\infty}^{\infty} f(x,y)dx.$$

Then to show $\int\int f = 1$, we need to show $\int_{-\infty}^{\infty} f_1(x)dx = 1$ (or $\int_{-\infty}^{\infty} f_2(y)dy = 1$). Then f_1, f_2 are densities, in *one* dimension. If $f(x,y) = f_{X,Y}(x,y)$ is the *joint* density of *two* random variables X, Y, then $f_1(x)$ is the density $f_X(x)$ of X, $f_2(y)$ the density $f_Y(y)$ of Y (f_1, f_2, or f_X, f_Y, are called the *marginal* densities of the *joint* density f, or $f_{X,Y}$).

To perform the integrations, we have to *complete the square*. We have the algebraic identity

$$(1-\rho^2)Q \equiv \Big[\Big(\frac{y-\mu_2}{\sigma_2}\Big) - \rho\Big(\frac{x-\mu_1}{\sigma_1}\Big)\Big]^2 + (1-\rho^2)\Big(\frac{x-\mu_1}{\sigma_1}\Big)^2$$

(reducing the number of occurrences of y to 1, as we intend to integrate out y first). Then (taking the terms free of y out through the y-integral)

$$f_1(x) = \frac{\exp(-\frac{1}{2}(x-\mu_1)^2/\sigma_1^2)}{\sigma_1\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sigma_2\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(\frac{-\frac{1}{2}(y-c_x)^2}{\sigma_2^2(1-\rho^2)}\right) dy, \tag{$*$}$$

where

$$c_x := \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1).$$

The integral is 1 ('normal density'). So

$$f_1(x) = \frac{\exp(-\frac{1}{2}(x-\mu_1)^2/\sigma_1^2)}{\sigma_1\sqrt{2\pi}},$$

which integrates to 1 ('normal density'), proving
Fact 1. $f(x,y)$ is a joint density function (two-dimensional), with marginal density functions $f_1(x), f_2(y)$ (one-dimensional). So we can write

$$f(x,y) = f_{X,Y}(x,y), \qquad f_1(x) = f_X(x), \qquad f_2(y) = f_Y(y).$$

Fact 2. X, Y are normal: X is $N(\mu_1, \sigma_1^2)$, Y is $N(\mu_2, \sigma_2^2)$. For, we showed $f_1 = f_X$ to be the $N(\mu_1, \sigma_1^2)$ density above, and similarly for Y by symmetry.
Fact 3. $EX = \mu_1, EY = \mu_2, varX = \sigma_1^2, varY = \sigma_2^2$.

This identifies four out of the five parameters: two means μ_i, two variances σ_i^2. Next, recall the definition of conditional probability: $P(A|B) := P(A \cap B)/P(B)$. In the *discrete* case, if X, Y take possible values x_i, y_j with probabilities $f_X(x_i), f_Y(y_j)$, (X, Y) takes possible values (x_i, y_j) with probabilities $f_{X,Y}(x_i, y_j)$:

$$f_X(x_i) = P(X = x_i) = \Sigma_j P(X = x_i, Y = y_j) = \Sigma_j f_{X,Y}(x_i, y_j).$$

Then the *conditional* distribution of Y *given* $X = x_i$ is

$$f_{Y|X}(y_j|x_i) = P(Y = y_j \& X = x_i)/P(X = x_i) = f_{X,Y}(x_i, y_j)/\sum_j f_{X,Y}(x_i, y_j),$$

and similarly with X, Y interchanged.

In the *density* case, we have to replace *sums* by *integrals*:

$$f_{Y|X}(y|x) := f_{X,Y}(x, y)/f_X(x) = f_{X,Y}(x, y)/\int_{-\infty}^{\infty} f_{X,Y}(x, y)dy.$$

Returning to the bivariate normal:
Fact 4. The conditional distribution of y given $X = x$ is $N(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1), \quad \sigma_2^2(1 - \rho^2))$.
Proof. Go back to completing the square (or, return to (*) with $\int$ and dy deleted):

$$f(x, y) = \frac{\exp(-\frac{1}{2}(x - \mu_1)^2/\sigma_1^2)}{\sigma_1\sqrt{2\pi}} . \frac{\exp(-\frac{1}{2}(y - c_x)^2/(\sigma_2^2(1 - \rho^2)))}{\sigma_2\sqrt{2\pi}\sqrt{1 - \rho^2}}.$$

The first factor is $f_1(x)$, by Fact 1. So, $f_{Y|X}(y|x) = f(x, y)/f_1(x)$ is the second factor:

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1 - \rho^2}} \exp\Big(\frac{-(y - c_x)^2}{2\sigma_2^2(1 - \rho^2)}\Big),$$

where c_x is the linear function of x given below (*). //

This not only completes the proof of Fact 4 but gives
Fact 5. The conditional mean $E(Y|X = x)$ is *linear* in x:

$$E(Y|X = x) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1).$$

NHB