livsoln5.tex

SOLUTIONS 5. 3.3.2014

Q1. That the conditional variance of Y given $X = x$ is

$$var(Y|X = x) = \sigma_2^2(1 - \rho^2).$$

follows from Prob4 Q4.

Recall (Prob4 Q3) that the variability (= variance) of Y is $varY = \sigma_2^2$. By Prob4 Q5, the variability remaining in Y when X is given (i.e., not accounted for by knowledge of X) is $\sigma_2^2(1-\rho^2)$. Subtracting: the variability of Y which *is* accounted for by knowledge of X is $\sigma_2^2\rho^2$. That is: ρ^2 is the *proportion of the variability* of Y accounted for by knowledge of X. So ρ is a measure of the *strength of association* between Y and X.

Recall that the *covariance* is defined by

$$\begin{aligned} cov(X,Y) &:= E[(X - EX)(Y - EY)] = E[(X - \mu_1)(Y - \mu_2)], \\ &= E(XY) - (EX)(EY), \end{aligned}$$

and the *correlation coefficient* ρ, or $\rho(X, Y)$, defined by

$$\rho = \rho(X,Y) := cov(X,Y)/(\sqrt{varX}\sqrt{varY}) = E[(X - \mu_1)(Y - \mu_2)]/\sigma_1\sigma_2$$

is the usual measure of the strength of association between X and Y ($-1 \leq \rho \leq 1$; $\rho = \pm 1$ iff one of X, Y is a function of the other).

Q2. The correlation coefficient of X, Y is ρ.
Proof.

$$\rho(X,Y) := E\Big[\Big(\frac{X - \mu_1}{\sigma_1}\Big)\Big(\frac{Y - \mu_2}{\sigma_2}\Big)\Big] = \int\int\Big(\frac{x - \mu_1}{\sigma_1}\Big)\Big(\frac{y - \mu_2}{\sigma_2}\Big)f(x,y)dxdy.$$

Substitute for $f(x, y) = c\exp(-\frac{1}{2}Q)$, and make the change of variables $u := (x - \mu_1)/\sigma_1$, $v := (y - \mu_2)/\sigma_2$:

$$\rho(X,Y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\int\int uv\exp\Big(\frac{-[u^2 - 2\rho uv + v^2]}{2(1-\rho^2)}\Big)dudv.$$

Completing the square, $[u^2 - 2\rho uv + v^2] = (v - \rho u)^2 + (1 - \rho^2)u^2$. So

$$\rho(X,Y) = \frac{1}{\sqrt{2\pi}} \int u \exp\left(-\frac{u^2}{2}\right) du . \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \int v \exp\left(-\frac{(v-\rho u)^2}{2(1-\rho^2)}\right) dv.$$

Replace v in the inner integral by $(v - \rho u) + \rho u$, and calculate the two resulting integrals separately. The first is zero ('normal mean', or symmetry), the second is ρu ('normal density'). So

$$\rho(X,Y) = \frac{1}{\sqrt{2\pi}} . \rho \int u^2 \exp\left(-\frac{u^2}{2}\right) du = \rho$$

('normal variance'), as required. //

This completes the identification of all five parameters in the bivariate normal distribution: two means μ_i, two variances σ_i^2, one correlation ρ.

Q3. The bivariate normal law has *elliptical contours*.
Proof. The contours are $Q(x,y) = const$, which are ellipses.

Q4.

$$\begin{aligned} M(t_1,t_2) &= E(\exp(t_1 X + t_2 Y)) = \int\int \exp(t_1 x + t_2 y) f(x,y) dx dy \\ &= \int \exp(t_1 x) f_1(x) dx \int \exp(t_2 y) f(y|x) dy. \end{aligned}$$

The inner integral is the MGF of $Y|X = x$, which is $N(c_x, \sigma_2^2, (1-\rho^2))$, so is $\exp(c_x t_2 + \frac{1}{2}\sigma_2^2(1-\rho^2)t_2^2)$. By Prob4 Q5 $c_x t_2 = [\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1)]t_2$, so

$$M(t_1,t_2) = \exp(t_2\mu_2 - t_2\frac{\sigma_2}{\sigma_1}\mu_1 + \frac{1}{2}\sigma_2^2(1-\rho^2)t_2^2) \int \exp([t_1 + t_2\rho\frac{\sigma_2}{\sigma_1}]x) f_1(x) dx.$$

Since $f_1(x)$ is $N(\mu_1, \sigma_1^2)$, the inner integral is a normal MGF, which is thus $\exp(\mu_1[t_1 + t_2\rho\frac{\sigma_2}{\sigma_1}] + \frac{1}{2}\sigma_1^2[\ldots]^2)$. Combining and simplifying, we obtain

Q5. X, Y are independent if and only if $\rho = 0$.

Proof. For densities: X, Y are independent iff the joint density $f_{X,Y}(x,y)$ *factorises* as the *product* of the marginal densities $f_X(x).f_Y(y)$

For MGFs: X, Y are independent iff the joint MGF $M_{X,Y}(t_1,t_2)$ *factorises* as the *product* of the marginal MGFs $M_X(t_1).M_Y(t_2)$. From Q4, this occurs iff $\rho = 0$. Similarly with CFs, if we prefer to work with them. // NHB