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Proof of the Black-Scholes PDE (continued).

Substituting the values above (Week 9) in the no-arbitrage relation gives

$$\frac{-SF_2}{F - SF_2}.\mu + \frac{F}{F - SF_2}.\frac{F_1 + \mu S F_2 + \frac{1}{2}\sigma^2 F_{22}}{F} = r.$$

So

$$-SF_2\mu + F_1 + \mu S F_2 + \frac{1}{2}\sigma^2 S^2 F_{22} = rF - rSF_2,$$

giving

$$F_1 + rSF_2 + \frac{1}{2}\sigma^2 S^2 F_{22} - rF = 0. \qquad (BS)$$

This completes the proof of the Black-Scholes PDE. //

Corollary. The no-arbitrage price of the derivative does not depend on the mean return $\mu(t, .)$ of the underlying asset, only on its *volatility* $\sigma(t, .)$ and the short interest-rate.

The Black-Scholes PDE may be solved analytically, or numerically. We give an alternative probabilistic approach below.

The Black-Scholes PDE is parabolic, and can be transformed into the heat equation, whose solution can be written down in terms of an integral and the *heat kernel*. This is the same as the probabilistic solution obtained in Ch. IV (by taking the limit of the discrete-time BS *formula*), and again below (by continuous-time methods).

Note. 1. Black and Scholes were classically trained applied mathematicians. When they derived their PDE, they recognised it as parabolic. After some months' work, they were able to transform it into the heat equation. The solution to this is known classically.[1] On transforming back, they obtained the Black-Scholes formula.

The Black-Scholes formula transformed the financial world. Before it (see

[1]See e.g. the link to MPC2 (Mathematics and Physics for Chemists, Year 2) on my website, Weeks 4, 9. The solution is in terms of *Green functions*. The Green function for (fundamental solution of) the heat equation has the form of a normal density. This reflects the close link between the mathematics of the heat equation (J. Fourier (1768-1830) in 1807; *Théorie analytique de la chaleur* in 1822) and the mathematics of Brownian motion, which as we have seen belongs to the 20th Century. The link was made by S. Kakutani in 1944, and involves potential theory.

Ch. I), the expert view was that asking what an option is worth was (in effect) a silly question: the answer would necessarily depend on the attitude to risk of the individual considering buying the option. It turned out that – at least approximately (i.e., subject to the restrictions to perfect – frictionless – markets, including No Arbitrage – an over-simplification of reality) there *is* an option value. One can see this in one's head, without doing any mathematics, if one knows that the Black-Scholes market is *complete* (see VI.3, VI.4 below). So, every contingent claim (option, etc.) can be *replicated*, in terms of a suitable combination of cash and stock. Anyone can price this:
(i) count the cash, and count the stock;
(ii) look up the current stock price;
(iii) do the arithmetic.
2. The programmable pocket calculator was becoming available around this time. Every trader immediately got one, and programmed it, so that he could price an option (using the Black-Scholes model!) in real time, from market data.
3. The missing quantity in the Black-Scholes formula is the *volatility*, σ. But, the price is continuous and strictly increasing in σ (options like volatility!). So there is *exactly one* value of σ that gives the price at which options are being currently traded. The conclusion is that this is the value that the market currently judges σ to be. This is the value (called the *implied volatility* that traders use.
4. Because the Black-Scholes model is the benchmark model of mathematical finance, and gives a value for σ at the push of a button, it is widely used.
5. This is *despite* the fact that no one actually believes the Black-Scholes model! It gives at best an over-simplified approximation to reality. Indeed, Fischer Black himself famously once wrote a paper called *The holes in Black-Scholes*.
6. This is an interesting example of theory and practice interacting!
7. Black and Scholes has considerable difficulty in getting their paper published! It was ahead of its time. When published, and its importance understood, it changed its times.
8. Black-Scholes theory and its developments, plus the internet (a global network of fibre-optic cables – using *photons* rather than *electrons*), were important contributory factors to *globalization*. Enormous sums of money can be transported round the world at the push of a button, and are every day. This has led to *financial contagion* – "one country's economic problem becomes the world's economic problem". (The Ebola virus comes to mind here.)

§3. The Feynman-Kac Formula, Risk-Neutral Valuation and the Continuous Black-Scholes Formula

Suppose we consider a SDE, with initial condition (IC), of the form

$$dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s \qquad (t \le s \le T), \tag{SDE}$$

$$X_t = x. \tag{IC}$$

For suitably well-behaved functions μ, σ, this SDE has a unique solution $X = (X_s : t \le s \le T)$, a diffusion. We refer for details on solutions of SDEs and diffusions to an advanced text such as [RW2], [RY], [KS §5.7]. Uniqueness of solutions of the SDE is related to *completeness*, and uniqueness of prices: see VI.4 below. This is much as in the FTAP of Ch. IV, but the continuous-time case is harder – we have to quote uniqueness rather than prove it as we did there.

Taking existence of a unique solution for granted for the moment, consider a smooth function $F(s, X_s)$ of it. By Itô's Lemma,

$$dF = F_1 ds + F_2 dX + \frac{1}{2}F_{22}(dX)^2,$$

and as $(dX)^2 = (\mu ds + \sigma dW_s)^2 = \sigma^2(dW_s)^2 = \sigma^2 ds$, this is

$$dF = F_1 ds + F_2(\mu ds + \sigma dW_s) + \frac{1}{2}\sigma^2 F_{22} ds = (F_1 + \mu F_2 + \frac{1}{2}\sigma^2 F_{22})ds + \sigma F_2 dW_s. \tag{$*$}$$

Now suppose that F satisfies the PDE, with boundary condition (BC),

$$F_1(t, x) + \mu(t, x)F_2(t, x) + \frac{1}{2}\sigma^2 F_{22}(t, x) = g(t, x) \tag{PDE}$$

$$F(T, x) = h(x). \tag{BC}$$

Then $(*)$ gives

$$dF = g ds + \sigma F_2 dW_s,$$

which can be written in stochastic-integral form as

$$F(T, X_T) = F(t, X_t) + \int_t^T g(s, X_s)ds + \int_t^T \sigma(s, X_s)F_2(s, X_s)dW_s.$$

The stochastic integral on the right is a martingale, so has constant expectation, which must be 0 as it starts at 0. Recalling that $X_t = x$, writing $E_{t,x}$

for expectation with value x and starting-time t, and the price at expiry T as $h(X_T)$ as before, taking $E_{t,x}$ gives

$$E_{t,x}h(X_T) = F(t,x) + E_{t,x}\int_t^T g(s, X_s)ds.$$

This gives:

Theorem (Feynman-Kac Formula). The solution $F = F(t,x)$ to the PDE

$$F_1(t,x) + \mu(t,x)F_2(t,x) + \frac{1}{2}\sigma^2(t,x)F_{22}(t,x) = g(t,x) \qquad (PDE)$$

with final condition $F(T,x) = h(x)$ has the stochastic representation

$$F(t,x) = E_{t,x}h(X_T) - E_{t,x}\int_t^T g(s, X_s)ds, \qquad (FK)$$

where X satisfies the SDE

$$dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s \qquad (t \leq s \leq T) \qquad (SDE)$$

with initial condition $X_t = x$.

Now replace $\mu(t,x)$ by rx, $\sigma(t,x)$ by σx, g by rF in the Feynman-Kac formula above. The SDE becomes

$$dX_s = rX_s ds + \sigma X_s dW_s \qquad (**)$$

– the same as for a risky asset with mean return-rate r (the short interest-rate for a riskless asset) in place of μ (which disappeared in the Black-Scholes result). The PDE becomes

$$F_1 + rxF_2 + \frac{1}{2}\sigma^2 x^2 F_{22} = rF, \qquad (BS)$$

the Black-Scholes PDE. So by the Feynman-Kac formula,

$$dF = rFds + \sigma F_2 dW_s, \qquad F(T,s) = h(s).$$

We can eliminate the first term on the right by discounting at rate r: write $G(s, X_s) := e^{-rs}F(s, X_s)$ for the discounted price process. Then as before,

$$dG = -re^{-rs}Fds + e^{-rs}dF = e^{-rs}(dF - rFds) = e^{-rs}.\sigma F_2 dW.$$

Then integrating, G is a stochastic integral, so a martingale: *the discounted price process* $G(s, X_s) = e^{-rs} F(s, X_s)$ *is a martingale*, under the measure P^* giving the dynamics in $(**)$. This is the measure P we started with, *except* that μ has been changed to r. Thus, G has constant P^*-expectation:

$$E^*_{t,x} G(t, X_t) = E^*_{t,x} e^{-rt} F(t, X_t) = e^{-rt} F(t, x) = E^*_{T,x} e^{-rT} F(T, X_T) = e^{-rT} h(X_T).$$

This gives the following result:

Theorem (Risk-Neutral Valuation Formula). The no-arbitrage price of the claim $h(S_T)$ is given by

$$F(t, x) = e^{-r(T-t)} E^*_{t,x} h(S_T),$$

where $S_t = x$ is the asset price at time t and P^* is the measure under which the asset price dynamics are given by

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Corollary. In the Black-Scholes model above, the arbitrage-free price does not depend on the mean return rate μ of the underlying asset.

Comments.
1. *Risk-neutral measure.* We call P^* the *risk-neutral* probability measure. It is equivalent to P (by Girsanov's Theorem – the change-of-measure result, which deals with change of drift in SDEs – see §4), and is a martingale measure (as the discounted asset prices are P^*-martingales, by above), i.e. P^* (or Q) is the *equivalent martingale measure (EMM).*
2. *Fundamental Theorem of Asset Pricing.* The above continuous-time result may be summarised just as the Fundamental Theorem of Asset Pricing in discrete time: to get the no-arbitrage price of a contingent claim, take the discounted expected value under the equivalent mg (risk-neutral) measure.
3. *Completeness.* In discrete time, we saw that absence of arbitrage corresponded to *existence* of risk-neutral measures, completeness to *uniqueness*. We have obtained existence and uniqueness here (and so completeness), by appealing to existence and uniqueness theorems for PDEs (which we have not proved!). A more probabilistic route is to use Girsanov's Theorem (§4) instead. Completeness questions then become questions on representation theorems for Brownian martingales (§4). As usual, there is a choice of routes

to the major results – in this case, a trade-off between analysis (PDEs) and probability (Girsanov's Theorem and the Representation Theorem for Brownian Martingales, §4 below).

Now the process specified under P^* by the dynamics $(**)$ is our old friend geometric Brownian motion, $GBM(r,\sigma)$. Thus if S_t has P^*-dynamics

$$dS_t = rS_t dt + \sigma S_t dW_t, \qquad S_t = s,$$

with W a P^*-Brownian motion, then we can write S_T explicitly as

$$S_T = s\exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)\}.$$

Now $W_T - W_t$ is normal $N(0, T-t)$, so $(W_T - W_t)/\sqrt{T-t} =: Z \sim N(0,1)$:

$$S_T = s\exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma Z\sqrt{T-t}\}, \qquad Z \sim N(0,1).$$

So by the Risk-Neutral Valuation Formula, the pricing formula is

$$F(t,x) = e^{-r(T-t)}\int_{-\infty}^{\infty} h(s\exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(T-t)^{\frac{1}{2}}x\}).\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx.$$

For a general payoff function h, there is no explicit formula for the integral, which has to be evaluated numerically. But we can evaluate the integral for the basic case of a European call option with strike-price K:

$$h(s) = (s-K)^+.$$

Then

$$F(t,x) = e^{-r(T-t)}\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}[s\exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(T-t)^{\frac{1}{2}}x\} - K]_+ dx.$$

We have already evaluated integrals of this type in Chapter IV, where we obtained the Black-Scholes formula from the binomial model by a passage to the limit. Completing the square in the exponential as before gives the

Continuous Black-Scholes Formula.

$$F(t,s) = s\Phi(d_+) - e^{-r(T-t)}K\Phi(d_-), \qquad d_\pm := [\log(s/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{T-t}.$$

§4. Girsanov's Theorem

As above: by the Risk-Neutral Valuation Formula, to calculate option prices one
(i) discounts everything;
(ii) takes conditional expectations under the equivalent martingale measure (EMM), or risk-neutral measure – the measure (P^* or Q) equivalent to P under which *discounted asset prices become martingales*. This is a *change of measure*, and mathematically it has the effect of replacing the return rate μ on the risky stock by the riskless return rate r. We derived it by the historical route: the Black-Scholes PDE (§2) and the Feynman-Kac formula (§3). One can replace these two results by one, and avoid the analytical detour via PDEs, by using instead the next result – Girsanov's theorem.

Consider first ([KS], §3.5) independent $N(0,1)$ random variables $Z_1, \cdots, Z_n$ on $(\Omega, \mathcal{F}, \mathcal{P})$. Given a vector $\mu = (\mu_1, \cdots, \mu_n)$, consider a new probability measure $\tilde{P}$ on $(\Omega, \mathcal{F})$ defined by

$$\tilde{P}(d\omega) = \exp\{\Sigma_1^n \mu_i Z_i(\omega) - \frac{1}{2}\Sigma_1^n \mu_i^2\}.P(d\omega).$$

This is a positive measure as $\exp\{.\} > 0$, and integrates to 1 as $\int \exp\{\mu_i Z_i\} dP = \exp\{\frac{1}{2}\mu_i^2\}$, so is a probability measure. It is also equivalent to P (has the same null sets – actually, the only null set are Lebesgue-null sets, in each case), again as the exponential term is positive. Also

$$\begin{aligned}
\tilde{P}(Z_i \in dz_i, \quad i = 1, \cdots, n) &= \exp\{\Sigma_1^n \mu_i z_i - \frac{1}{2}\Sigma_1^n \mu_i^2\}.P(Z_i \in dz_i, \quad i = 1, \cdots, n) \\
&= (2\pi)^{-\frac{1}{2}n} \exp\{\Sigma \mu_i z_i - \frac{1}{2}\Sigma \mu_i^2 - \frac{1}{2}\Sigma z_i^2\}\Pi dz_i \\
&= (2\pi)^{-\frac{1}{2}n} \exp\{-\frac{1}{2}\Sigma(z_i - \mu_i)^2\} dz_1 \cdots dz_n.
\end{aligned}$$

This says that if the Z_i are independent $N(0,1)$ under P, they are independent $N(\mu_i, 1)$ under $\tilde{P}$. Thus the effect of the *change of measure* $P \to \tilde{P}$, from the original measure P to the *equivalent* measure $\tilde{P}$, is to *change the mean*, from $0 = (0, \cdots, 0)$ to $\mu = (\mu_1, \cdots, \mu_n)$.

This result extends to infinitely many dimensions – i.e., from random vectors to stochastic processes, indeed with random rather than deterministic means. We quote (Igor Vladimirovich GIRSANOV (1934-67) in 1960):

Theorem (Girsanov's Theorem). Let $(\mu_t : 0 \le t \le T)$ be an adapted (e.g., left-continuous) process with $\int_0^T \mu_t^2 dt < \infty \quad a.s.$, and such that the process $(L_t : 0 \le t \le T)$ defined by

$$L_t = \exp\{-\int_0^t \mu_s dW_s - \frac{1}{2}\int_0^t \mu_s^2 ds\}$$

is a martingale. Then, under the probability P_L with density L_T relative to P, the process W^* defined by

$$W_t^* := W_t + \int_0^t \mu_s ds, \qquad (0 \le t \le T)$$

is a standard Brownian motion.

Here, L_t is the Radon-Nikodym derivative of P_L w.r.t. P on the σ-algebra $\mathcal{F}_t$. In particular, for $\mu_t \equiv \mu$, change of measure by introducing the RN derivative $\exp\{\mu W_t - \frac{1}{2}\mu^2\}$ corresponds to a change of drift from 0 to μ.

Girsanov's Theorem (or the Cameron-Martin-Girsanov Theorem) is formulated in varying degrees of generality, and proved, in [KS, §3.5], [RY, VIII].

Consider now the Black-Scholes model, with dynamics

$$dB_t = rB_t dt, \qquad dS_t = \mu S_t dt + \sigma S_t dW_t.$$

The discounted asset prices $\tilde{S}_t := e^{-rt} S_t$ have dynamics given, as before, by

$$d\tilde{S}_t = -re^{-rt} S_t dt + e^{-rt} dS_t = -r\tilde{S}_t dt + \mu\tilde{S}_t dt + \sigma\tilde{S}_t dW_t$$

$$= (\mu - r)\tilde{S}_t dt + \sigma\tilde{S}_t dW_t.$$

Now the drift – dt – term here prevents $\tilde{S}_t$ being a martingale; the noise – dW_t – term gives a stochastic integral, which is a martingale. Girsanov's theorem suggests the change of measure from P to the EMM (or risk-neutral measure) P^* making the discounted asset price a martingale. This
(i) gives directly the continuous-time version of the Fundamental Theorem of Asset Pricing: *to price assets, take expectations of discounted prices under the risk-neutral measure* (see below for completeness and uniqueness of EMM and prices);

(ii) allows a probabilistic treatment of the Black-Scholes model, avoiding the detour via PDEs of §2, §3.

Theorem (Representation Theorem for Brownian Martingales). Let $(M_t : 0 \leq t \leq T)$ be a square-integrable martingale with respect to the Brownian filtration $(\mathcal{F}_t)$. Then there exists an adapted process $H = (H_t : 0 \leq t \leq T)$ with $E \int H_s^2 ds < \infty$ such that

$$M_t = M_0 + \int_0^t H_s dW_s, \qquad 0 \leq t \leq T.$$

That is, all Brownian martingales may be represented as stochastic integrals with respect to Brownian motion.

We refer to, e.g., [KS], [RY] for proof. The multidimensional version of the result also holds, and may be proved in the same way.

The economic relevance of the Representation Theorem is that it shows (see e.g. [KS, I.6]) that the Black-Scholes model is *complete* – that is, that equivalent martingale measures are unique, and so that *Black-Scholes prices are unique.* Mathematically, the result is purely a consequence of properties of the Brownian filtration. The desirable mathematical properties of Brownian motion are thus seen to have hidden within them desirable economic and financial consequences of real practical value.

To summarise the basic case (μ and σ constant) in a nutshell:
(i). Dynamics are given by GBM, $dS_t = \mu S dt + \sigma S dW_t$.
(ii). Discount: $d\tilde{S}_t = (\mu - r)\tilde{S} dt + \sigma \tilde{S} dW_t$.
(iii). Use Girsanov's Theorem to change μ to r: under P^*, $d\tilde{S}_t = \sigma \tilde{S} dW_t$.
(iv). Integrate: the RHS gives a P^*-martingale, so has constant E^*-expectation.
Comments.
1. Calculation. When solutions have to be found numerically (as is the case in general - though not for some important special cases such as European call options, considered below), we again have a choice of
(i) analytic methods: numerical solution of a PDE,
(ii) probabilistic methods: evaluation, by the Risk-Neutral Valuation Formula, of an expectation.
A comparison of convenience between these two methods depends on one's experience of numerical computation and the software available. However, in

the simplest case considered here, the probabilistic problem involves a one-dimensional integral, while the analytic problem is two-dimensional (involves a two-variable PDE: one variable would give an ODE!). So on dimensional grounds, and because of the probabilistic content of this course, we will generally prefer the probabilistic approach.
2. The Feynman-Kac formula. It is interesting to note that the Feynman-Kac formula originates in an entirely different context, namely quantum physics. In the late 1940s, the physicist Richard Feynman developed his path-integral approach to quantum mechanics, leading to his work (with Schwinger, Tomonaga and Dyson) on QED (quantum electrodynamics). Feynman's approach was non-rigorous; Mark Kac, an analyst and probabilist with an excellent background in PDE, produced a rigorous version which led to the approach above.
3. The Sharpe ratio. There is no point in investing in a risky asset with mean return rate μ, when cash is a riskless asset with return rate r, unless $\mu > r$. The excess return $\mu - r$ is compared with the risk, as measured by the volatility σ via the *Sharpe ratio*

$$\lambda := (\mu - r)/\sigma,$$

also known as the *market price of risk*.
4. The Greeks and delta-hedging. This is much as in discrete time (Ch. IV).
5. Discrete and continuous time. One often has a choice between discrete and continuous time. For discrete time, we have proved everything; for continuous time, we have had to quote the hard proofs. Note that in continuous time we can use calculus – PDEs, SDEs etc. In discrete time we use instead the calculus of finite differences.
6. The calculus of finite differences. This is very similar to ordinary calculus (old-fashioned name: the *infinitesimal calculus* – thus the opposite of finite here is infinitesimal, not infinite!). It is in some ways harder. For instance: you all know integration by parts (partial integration) backwards. The discrete analogue – partial summation, or Abel's lemma – may be less familiar.

The calculus of finite differences used to be taught for use in e.g. interpolation (how to use information in mathematical tables to 'fill in missing values'). This is now done by computer subroutines – *but*, computers work discretely (with differences rather than derivatives), so the subject is still alive and well.

§5. Infinite time-horizon; Real options (Investment options)

We sketch here the theory of the American option (one can exercise at any time), over an infinite time-horizon. We deal first with a *put* option (see Week 11 under *Real options* for the corresponding 'call option') – giving the right to sell at the strike price K, at any time τ of our choosing. This τ has to be a *stopping time*: we have to take the decision whether or not to stop at τ based on information already available (that is, contained in $\mathcal{F}_\tau$ – no access to the future, no insider trading). As above, we pass to the risk-neutral measure.

Under the risk-neutral measure, the SDE for GBM becomes

$$dX_t = rX_t dt + \sigma X_t dB_t. \tag{GBM_r}$$

To evaluate the option, we have to solve the *optimal stopping problem*

$$V(x) := \sup_\tau E_x[e^{-r\tau}(K - X_\tau)^+]$$

where the supremum is taken over all stopping times τ and $X_0 = x$ under P_x.

The process X satisfying (GBM_r) is specified by a second-order linear differential operator, called its (infinitesimal) *generator*,

$$L_X := rxD + \frac{1}{2}\sigma^2 x^2 D^2, \qquad D := \partial/\partial x.$$

Now the closer X gets to 0, the less likely we are to gain by continuing. This suggests that our best strategy is to stop when X gets too small: to stop at $\tau = \tau_b$, where

$$\tau_b := \inf\{t \geq 0 : X_t \leq b\},$$

for some $b \in (0, K)$. This gives the following *free boundary problem* for the *unknown value function* $V(x)$ and the *unknown point* b:

$$L_X V = rV \qquad \text{for } x > b; \tag{i}$$

$$V(x) = (K - x)^+ \qquad \text{for } x = b; \tag{ii}$$

$$V'(x) = -1 \qquad \text{for } x = b \text{ (smooth fit)}; \tag{iii}$$

$$V(x) > (K - x)^+ \qquad \text{for } x > b; \tag{iv}$$

$$V(x) = (K - x)^+ \qquad \text{for } 0 < x < b. \tag{v}$$

Writing $d := \sigma^2/2$ ('d for diffusion'), (i) is

$$dx^2V'' + rxV' - rV = 0. \tag{i^*}$$

Trial solution:

$$V(x) = x^p.$$

Substituting gives a quadratic for p:

$$p^2 - (1 - \frac{r}{d})p - \frac{r}{d} = 0.$$

One root is $p = 1$; the other is $p = -r/d$. So the general solution (GS) to the DE (i^*) is

$$V(x) = C_1x + C_2x^{-r/d},$$

for some constants C_1 and C_2. But $V(x) \leq K$ for all $x \geq 0$ (an option giving the right to sell at price K cannot be worth more than K!), so $C_1 = 0$. This gives

$$C_2 = \frac{d}{r}\Big(\frac{K}{1 + d/r}\Big)^{1+r/d}, \qquad b = \frac{K}{1 + d/r}.$$

So

$$\begin{aligned} V(x) &= \frac{d}{r}\Big(\frac{K}{1 + d/r}\Big)^{1+r/d} x^{-r/d} \qquad \text{if } x \in [b, \infty) \\ &= K - x \qquad \text{if } x \in (0, b]. \end{aligned}$$

This is in fact the full and correct solution to the problem. For details, see [P&S], §25.1.

The 'smooth fit' in (iii) is characteristic of free boundary problems. For a heuristic analogy: imagine trying to determine the shape of a rope, tied to the ground on one side of a convex body, stretched over the body, then pulled tight and tied to the ground on the other side. We can see on physical grounds that the rope will be:
straight to the left of the convex body;
continuously in contact with the body for a while, then
straight to the right of the body, and
there should be no kink in the rope at the points where it makes and then leaves contact with the body.
This 'no kink' condition corresponds to 'smooth fit' in (iii).