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Approximation (continued).

It is not possible to include detailed proofs of these assertions in a course of this type [recall that we did not construct the measure-theoretic integral of Ch. II in detail either - and this is harder!]. The key technical ingredient needed is the *Kunita-Watanabe inequalities*. See e.g. [KS], §§3.1-2.

One can define stochastic integration in much greater generality.

1. *Integrands.* The natural class of integrands X to use here is the class of *predictable* processes. These include the left-continuous processes to which we confine ourselves above.

2. *Integrators.* One can construct a closely analogous theory for stochastic integrals with the Brownian integrator B above replaced by a *continuous local martingale* integrator M (or more generally by a *local martingale*: see below). The properties above hold, with D replaced by

$$E[(\int_0^t X_u dM_u)^2] = E \int_0^t X_u^2 d\langle M\rangle_u.$$

See e.g. [KS], [RY] for details.

One can generalise further to *semimartingale* integrators: these are processes expressible as the sum of a local martingale and a process of (locally) finite variation. Now C is replaced by: stochastic integrals of local martingales are local martingales. See e.g. [RW1] or Meyer (1976) for details.

§6. Stochastic Differential Equations (SDEs) and Itô's Lemma

Suppose that U, V are adapted processes,with U locally integrable (so $\int_0^t U_s ds$ is defined as an ordinary integral, as in Ch. II), and V is left-continuous with $\int_0^t EV_u^2 du < \infty$ for all t (so $\int_0^t V_s dB_s$ is defined as a stochastic integral, as in §5). Then

$$X_t := x_0 + \int_0^t U_s ds + \int_0^t V_s dB_s$$

defines a stochastic process X with $X_0 = x_0$. It is customary, and convenient, to express such an equation symbolically in differential form, in terms of the *stochastic differential equation*

$$dX_t = U_t dt + V_t dB_t, \qquad X_0 = x_0. \qquad (SDE)$$

Now suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a function, continuously differentiable once in its first argument (which will denote time), and twice in its second argument (space): $f \in C^{1,2}$. The question arises of giving a meaning to the stochastic differential $df(t, X_t)$ of the process $f(t, X_t)$, and finding it.

Recall the Taylor expansion of a smooth function of several variables, $f(x_0, x_1, \cdots, x_d)$ say. We use suffices to denote partial derivatives: $f_i := \partial f/\partial x_i$, $f_{i,j} := \partial^2 f/\partial x_i \partial x_j$ (recall that if partials not only exist but are continuous, then the order of partial differentiation can be changed: $f_{i,j} = f_{j,i}$, etc. – *Clairault's theorem*). Then for $x = (x_0, x_1, \cdots, x_d)$ near u,

$$f(x) = f(u) + \Sigma_{i=0}^d (x_i - u_i) f_i(u) + \frac{1}{2} \Sigma_{i,j=0}^d (x_i - u_i)(x_j - u_j) f_{i,j}(u) + \cdots$$

In our case (writing t_0 in place of 0 for the starting time):

$$f(t, X_t) = f(t_0, X(t_0)) + (t - t_0) f_1(t_0, X(t_0)) + (X(t) - X(t_0)) f_2 + \frac{1}{2}(t - t_0)^2 f_{11} +$$

$$(t - t_0)(X(t) - X(t_0)) f_{12} + \frac{1}{2}(X(t) - X(t_0))^2 f_{22} + \cdots,$$

which may be written symbolically as

$$df(t, X(t)) = f_1 dt + f_2 dX + \frac{1}{2} f_{11} (dt)^2 + f_{12} dt dX + \frac{1}{2} f_{22} (dX)^2 + \cdots.$$

In this, we
(i) substitute $dX_t = U_t dt + V_t dB_t$ from above,
(ii) substitute $(dB_t)^2 = dt$, i.e. $|dB_t| = \sqrt{dt}$, from §4:

$$df = f_1 dt + f_2 (U dt + V dB) + \frac{1}{2} f_{11} (dt)^2 + f_{12} dt (U dt + V dB) + \frac{1}{2} f_{22} (U dt + V dB)^2 + \cdots$$

Now using $(dB)^2 = dt$,

$$\begin{aligned} (U dt + V dB)^2 &= V^2 dt + 2UV dt dB + U^2 (dt)^2 \\ &= V^2 dt + \text{higher-order terms}: \end{aligned}$$

$$df = (f_1 + U f_2 + \frac{1}{2} V^2 f_{22}) dt + V f_2 dB + \text{higher-order terms}.$$

Summarising, we obtain *Itô's Lemma*, the analogue for the Itô or stochastic calculus of the chain rule for ordinary (Newton-Leibniz) calculus:

Theorem (Itô's Lemma). If X_t has stochastic differential

$$dX_t = U_t dt + V_t dB_t, \qquad X_0 = x_0,$$

and $f \in C^{1,2}$, then $f = f(t, X_t)$ has stochastic differential

$$df = (f_1 + U f_2 + \frac{1}{2} V^2 f_{22}) dt + V f_2 dB_t.$$

That is, writing f_0 for $f(0, x_0)$, the initial value of f,

$$f(t, X_t)) = f_0 + \int_0^t (f_1 + U f_2 + \frac{1}{2} V^2 f_{22}) dt + \int_0^t V f_2 dB.$$

This important result may be summarised as follows: use Taylor's theorem formally, together with the rule

$$(dt)^2 = 0, \qquad dtdB = 0, \qquad (dB)^2 = dt.$$

Itô's Lemma extends to higher dimensions, as does the rule above:

$$df = (f_0 + \Sigma_{i=1}^d U_i f_i + \frac{1}{2} \Sigma_1^d V_i^2 f_{ii}) dt + \Sigma_1^d V_i f_i dB_i$$

(where U_i, V_i, B_i denote the ith coordinates of vectors U, V, B, f_i, f_{ii} denote partials as above); here the formal rule is

$$(dt)^2 = 0, \qquad dtdB_i = 0, \qquad (dB_i)^2 = dt, \qquad dB_i dB_j = 0 \qquad (i \neq j).$$

Corollary. $Ef(t, X_t) = f_0 + \int_0^t E[f_1 + U f_2 + \frac{1}{2} V^2 f_{22}] dt$.

Proof. $\int_0^t V f_2 dB$ is a stochastic integral, so a martingale, so its expectation is constant (= 0, as it starts at 0). //

Note. Powerful as it is in the setting above, Itô's Lemma really comes into its own in the more general setting of semimartingales. It says there that if X is a semimartingale and f is a smooth function as above, then $f(t, X(t))$ is also a semimartingale. The ordinary differential dt gives rise to the finite-variation part, the stochastic differential gives rise to the martingale part. This closure property under very general non-linear operations is very powerful and important.

Example: The Ornstein-Uhlenbeck Process.

The most important example of a SDE for us is that for geometric Brownian motion (VI.1 below). We close here with another example.

Consider now a model of the velocity V_t of a particle at time t ($V_0 = v_0$), moving through a fluid or gas, which exerts
(i) a frictional drag, assumed propertional to the velocity,
(ii) a noise term resulting from the random bombardment of the particle by the molecules of the surrounding fluid or gas. The basic model is the SDE

$$dV = -\beta V dt + c dB, \tag{OU}$$

whose solution is called the *Ornstein-Uhlenbeck* (velocity) process with *relaxation time* $1/\beta$ and *diffusion coefficient* $D := \frac{1}{2}c^2/\beta^2$. It is a stationary Gaussian Markov process (not stationary-increments Gaussian Markov like Brownian motion), whose limiting (ergodic) distribution is $N(0, \beta D)$ and whose limiting correlation function is $e^{-\beta|\cdot|}$.

If we integrate the OU velocity process to get the OU *displacement process*, we lose the Markov property (though the process is still Gaussian). Being non-Markov, the resulting process is much more difficult to analyse.

The OU process is the prototype of processes exhibiting *mean reversion*, or a *central push*: frictional drag acts as a restoring force tending to push the process back towards its mean. It is important in many areas, including
(i) statistical mechanics, where it originated;
(ii) mathematical finance, where it appears in the *Vasicek model* for the term-structure of interest-rates (the mean represents the 'natural' interest rate);
(iii) *stochastic volatility* models, where the volatility σ itself is now a stochastic process σ_t, subject to an SDE of OU type.

Theory of interest rates.

This subject dominates the mathematics of *money market*, or *bond markets*. These are more important in today's world than stock markets, but are more complicated, so we must be brief here. The area is crucially important in *macro-economic policy*, and in political decision-making, particularly after the financial crisis ("credit crunch"). Government policy is driven by fear of speculators in the bond markets (rather than aimed at inter-governmental cooperation against them). The mathematics is infinite-dimensional (at each time-point t we have a whole *yield curve* over future times), but reduces to finite-dimensionality: bonds are only offered at discrete times, with a *tenor* structure (a finite set of maturity times).

Mean reversion is used in models, to reflect the underlying 'natural interest rate', from which deviations may occur due to short-term pressures (pre-Crash – these may be longer-lasting nowadays, as we see post-Crash).

Chapter VI. MATHEMATICAL FINANCE IN CONTINUOUS TIME

§1. Geometric Brownian Motion (GBM)

As before, we write B for standard Brownian motion. We write $B_{\mu,\sigma}$ for Brownian motion with *drift* μ and *diffusion coefficient* σ: the path-continuous Gaussian process with independent increments such that

$$B_{\mu,\sigma}(s+t) - B_{\mu,\sigma}(s) \text{ is } N(\mu t, \sigma^2 t).$$

This may be realised as

$$B_{\mu,\sigma}(t) = \mu t + \sigma B(t).$$

Consider the process

$$X_t = f(t, B_t) := x_0. \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\}.$$

Here, since

$$f(t,x) = x_0. \exp\{(\mu - \frac{1}{2}s^2)t + \sigma x\},$$

$$f_1 = (\mu - \frac{1}{2}\sigma^2)f, \qquad f_2 = \sigma f, \qquad f_{22} = \sigma^2 f.$$

By Itô's Lemma (Ch. V: $dX_t = U_t dt + V_t dB_t$ and f smooth implies $df = (f_1 + U f_2 + \frac{1}{2}V^2 f_{22})dt + V f_2 dB_t$) we have (taking $U = 0, \quad V = 1, \quad X = B$),

$$dX_t = df = [(\mu - \frac{1}{2}\sigma^2)f + \frac{1}{2}\sigma^2 f]dt + \sigma f dB_t :$$

$$dX_t = \mu f dt + \sigma f dB_t = \mu X_t dt + \sigma X_t dB_t :$$

X satisfies the SDE

$$dX_t = X_t(\mu dt + \sigma dB_t), \qquad (GBM)$$

and is called *geometric Brownian motion* (GBM). We turn to its economic meaning, and the role of the two parameters μ and σ, below.

We recall the model of Brownian motion from Ch. V. It was developed (by Brown, Einstein, Wiener, ...) in *statistical mechanics*, to model the irregular, random motion of a particle suspended in fluid under the impact of collisions with the molecules of the fluid.

The situation in *economics* and *finance* is analogous: the price of an asset depends on many factors (a share in a manufacturing company depends on, say, its own labour costs, and raw material prices for the articles it manufactures. Together, these involve, e.g., foreign exchange rates, labour costs – domestic and foreign, transport costs, etc. – all of which respond to the unfolding of events – economic data/political events/the weather/technological change/labour, commercial and environmental legislation/ ... in time. There is also the effect of individual transactions in the buying and selling of a traded asset on the asset price. The analogy between the buffeting effect of molecules on a particle in the statistical mechanics context on the one hand, and that of this continuous flood of new price-sensitive information on the other, is highly suggestive. The first person to use Brownian motion to model price movements in economics was Bachelier in his celebrated thesis of 1900.

Bachelier's seminal work was not definitive (indeed, not correct), either mathematically (it was pre-Wiener) or economically. In particular, Brownian motion itself is inadequate for modelling prices, as
(i) it attains negative levels, and
(ii) one should think in terms of *return*, rather than prices themselves.
However, one can allow for both of these by using *geometric*, rather than ordinary, Brownian motion as one's basic model. This has been advocated in economics from 1965 on by Samuelson[1] – and was Itô's starting-point for his development of Itô or stochastic calculus in 1944 – and has now become standard:
SAMUELSON, P. A. (1965): Rational theory of warrant pricing. *Industrial Management Review* **6**, 13-39,
SAMUELSON, P. A. (1973): Mathematics of speculative prices. *SIAM Review* **15**, 1-42.
Returning now to (GBM), the SDE above for geometric Brownian motion driven by Brownian noise, we can see how to interpret it. We have a risky asset (stock), whose price at time t is X_t; $dX_t = X(t+dt) - X(t)$ is the change in X_t over a small time-interval of length dt beginning at time t; dX_t/X_t is

[1]Paul A. Samuelson (1915-2009), American economist; Nobel Prize in Economics, 1970

the gain per unit of value in the stock, i.e. the *return*. This is a sum of two components:
(i) a deterministic component μdt, equivalent to investing the money risklessly in the bank at interest-rate μ (> 0 in applications), called the *underlying return rate* for the stock,
(ii) a random, or noise, component σdB_t, with *volatility* parameter $\sigma > 0$ and driving Brownian motion B, which models the market uncertainty, i.e. the effect of noise.
Justification. For a treatment of this and other diffusion models via microeconomic arguments, see
[FS] FÖLLMER, H. & SCHWEIZER, M. (1993): A microeconomic approach to diffusion models for stock prices. *Mathematical Finance* **3**, 1-23.
Note. Observe the decomposition of what we are modelling into two components: a systematic component and a random component (driving noise). We have met such decompositions elsewhere – e.g. regression, and the Doob decomposition.

§2. The Black-Scholes Model

For the purposes of this section only, it is convenient to be able to use the 'W for Wiener' notation for Brownian motion/Wiener process, thus liberating B for the alternative use 'B for bank [account]'. Thus our driving noise terms will now involve dW_t, our deterministic [bank-account] terms dB_t.

We now consider an investor constructing a trading strategy in continuous time, with the choice of two types of investment:
(i) riskless investment in a bank account paying interest at rate $r > 0$ (the *short rate* of interest): $B_t = B_0 e^{rt} \quad (t \geq 0)$ [we neglect the complications involved in possible failure of the bank – though *banks do fail* – witness Barings 1995, or AIB 2002!];
(ii) risky investment in stock, one unit of which has price modelled as above by $GMB(\mu, \sigma)$. Here the volatility $\sigma > 0$; the restriction $0 < r < \mu$ on the short rate r for the bank and underlying rate μ for the stock are economically natural (but not mathematically necessary); the stock dynamics are thus given by

$$dS_t = S_t(\mu dt + \sigma dW_t).$$

Notation. Later, we shall need to consider several types of risky stock – d stocks, say. It is convenient, and customary, to use a *superscript* i to label stock type, $i = 1, \cdots, d$; thus $S^1, \cdots, S^d$ are the risky stock prices. We can then use a superscript 0 to label the bank account, S^0. So with one risky asset as above (Week 9), the dynamics are

$$\begin{aligned} dS_t^0 &= rS_t^0 dt, \\ dS_t^1 &= \mu S_t^1 dt + \sigma S_t^1 dW_t. \end{aligned}$$

We shall focus on pricing at time 0 of options with expiry time T; thus the index-set for time t throughout may be taken as $[0, T]$ rather than $[0, \infty)$.

We proceed as in the discrete-time model of IV.1. A *trading strategy* H is a vector stochastic process

$$H = (H_t : 0 \leq t \leq T) = ((H_t^0, H_t^1, \cdots, H_t^d)) : 0 \leq t \leq T)$$

which is *previsible*: each H_t^i is a previsible process (so, in particular, $(\mathcal{F}_{t-})$-adapted) [we may simplify with little loss of generality by replacing previsibility here by *left-continuity* of H_t in t]. The vector $H_t = (H_t^0, H_t^1, \cdots, H_t^d)$ is the *portfolio* at time t. If $S_t = (S_t^0, S_t^1, \cdots, S_t^d)$ is the vector of *prices* at time t, the *value* of the portfolio at t is the scalar product

$$V_t(H) := H_t.S_t = \Sigma_{i=0}^d H_t^i S_t^i.$$

The *discounted value* is

$$\tilde{V}_t(H) = \beta_t(H_t.S_t) = H_t.\tilde{S}_t,$$

where $\beta_t := 1/S_t^0 = e^{-rt}$ (fixing the scale by taking the initial bank account as 1, $S_0^0 = 1$), so

$$\tilde{S}_t = (1, \beta_t S_t^1, \cdots, \beta_t S_t^d)$$

is the vector of discounted prices.

Recall that
(i) in IV.1 H is a self-financing strategy if $\Delta V_n(H) = H_n.\Delta S_n$, i.e. $V_n(H)$ is the martingale transform of S by H,
(ii) stochastic integrals are the continuous analogues of martingale transforms.
We thus define the strategy H to be *self-financing*, $H \in SF$, if

$$dV_t = H_t.dS_t = \Sigma_0^d H_t^i dS_t^i.$$

The discounted value process is

$$\tilde{V}_t(H) = e^{-rt} V_t(H)$$

and the interest rate is r. So

$$d\tilde{V}_t(H) = -re^{-rt} dt.V_t(H) + e^{-rt} dV_t(H)$$

(since e^{-rt} has finite variation, this follows from integration by parts,

$$d(XY)_t = X_t dY_t + Y_t dX_t + \frac{1}{2} d\langle X, Y \rangle_t$$

– the quadratic covariation of a finite-variation term with any term is zero)

$$\begin{aligned} &= -re^{-rt} H_t.S_t dt + e^{-rt} H_t.dS_t \\ &= H_t.(-re^{-rt} S_t dt + e^{-rt} dS_t) \\ &= H_t.d\tilde{S}_t \end{aligned}$$

($\tilde{S}_t = e^{-rt} S_t$, so $d\tilde{S}_t = -re^{-rt} S_t dt + e^{-rt} dS_t$ as above): for H self-financing,

$$dV_t(H) = H_t.dS_t, \qquad d\tilde{V}_t(H) = H_t.d\tilde{S}_t,$$

$$V_t(H) = V_0(H) + \int_0^t H_s dS_s, \qquad \tilde{V}_t(H) = \tilde{V}_0(H) + \int_0^t H_s d\tilde{S}_s.$$

Now write $U_t^i := H_t^i S_t^i / V_t(H) = H_t^i S_t^i / \Sigma_j H_t^j S_t^j$ for the *proportion* of the value of the portfolio held in asset $i = 0, 1, \cdots, d$. Then $\Sigma U_t^i = 1$, and $U_t = (U_t^0, \cdots, U_t^d)$ is called the *relative portfolio*. For H self-financing,

$$dV_t = H_t.dS_t = \Sigma H_t^i dS_t^i = V_t \Sigma \frac{H_t^i S_t^i}{V_t} . \frac{dS_t^i}{S_t^i} : \qquad dV_t = V_t \Sigma U_t^i dS_t^i / S_t^i.$$

Dividing through by V_t, this says that the return dV_t/V_t is the weighted average of the returns dS_t^i/S_t^i on the assets, weighted according to their proportions U_t^i in the portfolio.

Note. Having set up this notation (that of [HP]) – in order to be able if we wish to have a basket of assets in our portfolio – we now prefer – for simplicity – to specialise back to the simplest case, that of one risky asset. Thus we will now take $d = 1$ until further notice.

Arbitrage. This is as in discrete time: an admissible ($V_t(H) \geq 0$ for all t) self-financing strategy H is an *arbitrage* (strategy, or opportunity) if

$$V_0(H) = 0, \qquad V_T(H) > 0 \qquad \text{with positive } P\text{-probability.}$$

The market is *viable*, or *arbitrage-free*, or NA, if there are no arbitrage opportunities.

We see first that if the value-process V satisfies the SDE

$$dV_t(H) = K(t)V_t(H)dt$$

– that is, if there is no driving Wiener (or noise) term – then $K(t) = r$, the short rate of interest. For, if $K(t) > r$, we can *borrow* money from the bank at rate r and *buy* the portfolio. The value grows at rate $K(t)$, our debt grows at rate r, so our net profit grows at rate $K(t) - r > 0$ – an arbitrage. Similarly, if $K(t) < r$, we can *invest* money in the bank and *sell the portfolio short*. Our net profit grows at rate $r - K(t) > 0$, risklessly – again an arbitrage. We have proved the

Proposition. In an arbitrage-free (NA) market, a portfolio whose value process has no driving Wiener term in its dynamics must have return rate r, the short rate of interest.

We restrict attention to arbitrage-free (viable) markets from now on.

We now consider tradable derivatives, whose price at expiry depends only on $S(T)$ (the final value of the stock) – $h(S(T))$, say, and whose price Π_t depends smoothly on the asset price S_t: for some smooth function F,

$$\Pi_t := F(t, S_t).$$

The dynamics of the riskless and risky assets are

$$dB_t = rB_t dt, \qquad dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ, σ may depend on both t and S_t:

$$\mu = \mu(t, S_t), \qquad \sigma = \sigma(t, S_t).$$

The next result is the celebrated *Black-Scholes partial differential equation* (PDE) of 1973, one of the central results of the subject:

Theorem (Black-Scholes PDE). In a market with one riskless asset B_t and one risky asset S_t, with short interest-rate r and dynamics

$$\begin{aligned} dB_t &= rB_t dt, \\ dS_t &= \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t, \end{aligned}$$

let a contingent claim be tradable, with price $h(S_T)$ at expiry T and price process $\Pi_t := F(t, S_t)$ for some smooth function F. Then the only pricing function F which does not admit arbitrage is the solution to the Black-Scholes PDE with boundary condition:

$$F_1(t,x) + rxF_2(t,x) + \frac{1}{2}x^2\sigma^2(t,x)F_{22}(t,x) - rF(t,x) = 0, \qquad (BS)$$

$$F(T,x) = h(x). \qquad (BC)$$

Proof. By Itô's Lemma,

$$d\Pi_t = F_1 dt + F_2 dS_t + \frac{1}{2}F_{22}(dS_t)^2$$

(since t has finite variation, the F_{11}- and F_{12}-terms are absent as $(dt)^2$ and $dtdS_t$ are negligible with respect to the terms retained)

$$= F_1 dt + F_2(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}F_{22}(\sigma S_t dW_t)^2$$

(since the contribution of the finite-variation term in dt is negligible in the second differential, as above)

$$= (F_1 + \mu S_t F_2 + \frac{1}{2}\sigma^2 {S_t}^2 F_{22})dt + \sigma S_t F_2 dW_t$$

(as $(dW_t)^2 = dt$). Now $\Pi = F$, so

$$d\Pi_t = \Pi_t(\mu_\Pi(t)dt + \sigma_\Pi(t)dW_t),$$

where

$$\mu_\Pi(t) := (F_1 + \mu S_t F_2 + \frac{1}{2}\sigma^2 S_t^2 F_{22})/F, \qquad \sigma_\Pi(t) := \sigma S_t F_2/F.$$

Now form a portfolio based on two assets: the underlying stock and the option (recall that options are also assets in their own right – they have a value (Black-Scholes formula), and are traded (in large quantities)). Let the relative portfolio in stock S and derivative Π be (U_t^S, U_t^Π). Then the dynamics for the value V of the portfolio are given by

$$\begin{aligned} dV_t/V_t &= U_t^S dS_t/S_t + U_t^\Pi d\Pi_t/\Pi_t \\ &= U_t^S(\mu dt + \sigma dW_t) + U_t^\Pi(\mu_\Pi dt + \sigma_\Pi dW_t) \\ &= (U_t^S \mu + U_t^\Pi \mu_\Pi)dt + (U_t^S \sigma + U_t^\Pi \sigma_\Pi)dW_t, \end{aligned}$$

by above. Now both brackets are linear in U^S, U^Π, and $U^S + U^\Pi = 1$ as proportions sum to 1. This is one linear equation in the two unknowns U^S, U^Π, and we can obtain a second one by eliminating the driving Wiener term in the dynamics of V – for then, the portfolio is *riskless*, so must have return r by the Proposition, to avoid arbitrage. We thus solve the two equations

$$\begin{aligned} U^S + U^\Pi &= 1 \\ U^S\sigma + U^\Pi\sigma_\Pi &= 0. \end{aligned}$$

The solution of the two equations above is

$$U^\Pi = \frac{\sigma}{\sigma - \sigma_\Pi}, \qquad U^S = \frac{-\sigma_\Pi}{\sigma - \sigma_\Pi},$$

which as $\sigma_\Pi = \sigma S F_2/F$ gives the portfolio explicitly as

$$U^\Pi = \frac{F}{F - SF_2}, \qquad U^S = \frac{-SF_2}{F - SF_2}.$$

With this choice of relative portfolio, the dynamics of V are given by

$$dV_t/V = (U_t^S \mu + U_t^\Pi \mu_\Pi)dt,$$

which has no driving Wiener term. So, no arbitrage as above implies that the return rate is the short interest rate r:

$$U_t^S \mu + U_t^\Pi \mu_\Pi = r.$$

Now substitute the values (obtained above)

$$\mu_\Pi = (F + \mu S F_2 + \frac{1}{2}\sigma^2 S^2 F_{22})/F, \quad U^S = (-SF_2)/(F - SF_2), \quad U^\Pi = F/(F - SF_2).$$