livsoln6.tex

**SOLUTIONS 6. 16.3.2015**

Q1 *Doubling strategy.* (i) With $N$ the number of losses before the first win:

$$P(N=k) = P(L, L, \cdots, L(k \text{ times}), W) = (\frac{1}{2})^k.\frac{1}{2} = (\frac{1}{2})^{k+1}.$$

That is, $N$ is geometrically distributed with parameter 1/2. As

$$\sum_{k=0}^{\infty} P(N=k) = \sum_0^{\infty} (\frac{1}{2})^{k+1} = \frac{1}{2}/(1-\frac{1}{2}) = 1,$$

$P(N < \infty) = 1$: $N < \infty$ a.s. So one is certain to win eventually.
(ii) Let $S_n$ be one's fortune at time $n$. When $N = k$, one has losses at trials $1, 2, 3, \ldots, k$, with losses $1, 2, 4, \ldots, 2^{k-1}$, followed by a win at trial $k+1$ (of $2^k$). So one's fortune then is

$$2^k - (1 + 2 + 2^2 + \ldots + 2^{k-1}) = 2^k - (2^k - 1) = 1,$$

summing the finite geometric progression. So one's eventual fortune is $+1$ (which, by (i), one is certain to win eventually).
(iii) $N$ has PGF

$$P(s) := E[s^N] = \sum_{n=0}^{\infty} s^k P(N=k) = \sum_0^{\infty} s^k.(\frac{1}{2})^{k+1}$$

$$= \frac{1}{2}\sum_0^{\infty}(\frac{1}{2}s)^k = \frac{1}{2}/(1-\frac{1}{2}s) = 1/(2-s):$$

$$P'(s) = E[Ns^{N-1}] = (2-s)^{-2}; \qquad P'(1) = E[N] = 1.$$

So the mean time the game lasts is 1.
(iv) As with the simple random walk (Q2 below): this is an impossible strategy to use in reality, for two reasons:
(a) It depends on one's opponent's cooperation. What is to stop him trying this on you? If he does, the game degenerates into a simple coin toss, with the winner walking away with a profit of 1 (pound, or million pounds, say) – suicidally risky.

(b) Even with a cooperative opponent, it relies on the gambler having an unlimited amount of cash to bet with, or an unlimited line of credit – both hopelessly unrealistic in practice.

Q2 *First-passage time for simple random walk (SRW).*

Let $F(s) := s^T = \sum_1^\infty P(T = n)s^n = \sum_1^\infty f_n s^n$ be the PGF of $T$ $(= T_1$, the first passage time to 1). Since the first-passage time $T_2$ to 2 is the sum of the first-passage times from 0 to 1 (PGF $F$) and from 1 to 2 (PGF $F$ again), and these are independent (they involve disjoint blocks of independent tosses), $T_2$ has PGF $F_2(s) := E[s^{T_2}] = F(s)^2$.

Condition on the outcome $X_1$ of the first toss. If this is head $(+1)$, $T_1 = 1$. If it is a tail $(-1)$, $T = 1 + U$, where $U$, the first-passage time from $-1$ to 1, has PGF $F_2(s) = F(s)^2$ as above. So

$$F(s) := E[s^T] = E[s^T | X_1 = +1]P(X_1 = +1) + E[s^T | X_1 = -1]P(X_1 = -1)$$
$$= \frac{1}{2}.s + \frac{1}{2}.sF(s)^2$$

(as 1 has PGF $s$). So $F$ satisfies the quadratic

$$\frac{1}{2}sF(s)^2 - F(s) + \frac{1}{2}s = 0. \qquad \text{So} \qquad F(s) = \frac{1 \pm \sqrt{1-s^2}}{s}.$$

We need to take the $-$ sign here (as $F(s)$ contains no $s^{-1}$ term):

$$F(s) = \frac{1 - \sqrt{1-s^2}}{s}.$$

(i) Put $s = 1$: $F(1) = 1$, so $\sum_1^\infty P(T = n) = 1$, so $T < \infty$ a.s.

(ii)

$$F'(s) = -\frac{1}{s^2} + \frac{\sqrt{1-s^2}}{s} - \frac{1}{s}.\frac{\frac{1}{2}(-2s)}{\sqrt{1-s^2}} = -\frac{1}{s^2} + \frac{\sqrt{1-s^2}}{s} + \frac{1}{\sqrt{1-s^2}}.$$

So $F'(1) = E[T] = +\infty$.

(iii) In particular, $P(T = n) > 0$ for infinitely many $n$ (indeed, for all odd $n$). So no bound can be put on our maximum net loss before we realise our eventual gain.

This strategy is even more unrealistic than that in Q1: it has all the disadvantages there, plus another – infinite mean waiting time.

NHB