Math482 EXAMINATION 2015

6 questions; do 4; 25 marks each

Q1. We assume throughout that our markets are perfect, or frictionless. This entails a number of assumptions: mention the principal ones, and discuss each assumption briefly.

Q2. (i) Define the terms no arbitrage market, *complete* market, *equivalent martingale measure.*
(ii) State without proof the No Arbitrage Theorem, the Completeness Theorem, the Fundamental Theorem of Asset Pricing and the Risk-Neutral Valuation Formula.
(iii) Do real markets show arbitrage? If so, to what extent does this affect Black-Scholes theory?
(iv) Are real markets complete? If not, to what extent does this affect Black-Scholes theory?

Q3. The current price of rubber is \$ 185 per 100kg. Next year, the price will be up to 195 or down to 180, each with positive probability. Neglect interest.
(i) Price a call option C for 100kg of rubber next year, with strike price K the current price 185.
(ii) Hedge this option.
(iii) You see C being traded now for \$ 3. Explain how you would exploit any arbitrage opportunity present.
(iv) You see C being traded now for \$ 4. Explain how you would exploit any arbitrage opportunity present.
(v) Who buys such options, and why?
(vi) Who buys the corresponding put options, and why?

Q4. (i) Define (standard) Brownian motion $W = (W(t))$ or (W_t).
(ii) Explain briefly why Brownian motion occurs in mathematical finance as a model of the uncertainty, or 'noise', driving prices.
(iii) Find the covariance of Brownian motion.
(iv) Obtain the Brownian scaling property: show that for $c > 0$, scaled Brownian motion W_c, where $W_c(t) := W(c^2t)/c$, is again a Brownian motion.
(v) Comment briefly on the question of scale in mathematical finance, and so on the suitability of Brownian motion in the Black-Scholes model.

Q5. The *Ornstein-Uhlenbeck stochastic differential equation* is

$$dV = -\beta V dt + \sigma dW, \qquad (OU)$$

with W standard Brownian motion.
(i) Interpret (OU) physically.
(ii) Solve (OU) to obtain

$$V_t = v_0 e^{-\beta t} + \sigma e^{-\beta t} \int_0^t e^{\beta u} dW_u.$$

(iii) By using the Itô isometry, or otherwise, show that V_t has distribution

$$N(v_0 e^{-\beta t}, \sigma^2 (1 - e^{-2\beta t})/(2\beta)).$$

(iv) Using (iii) and the independence of Brownian increments, or otherwise, show that the covariance is

$$cov(V_t, V_{t+u}) = \sigma^2 e^{-\beta u}(1 - e^{-2\beta t})/(2\beta) \qquad (u \geq 0).$$

(v) Show that V is Markov.
(vi) What is the financial relevance of this model?

Q6. (i) Show how to formulate an American put option with an infinite time-horizon as an optimal-stopping problem.
(ii) Show further how to formulate this as a free-boundary problem.
(iii) Obtain the fundamental quadratic. Hence or otherwise, show how to obtain the option price.

N. H. Bingham