Math482 EXAMINATION 2015-16

Six questions; do four; twenty-five marks each.

Q1. Comment briefly on:
(i) size of trades;
(ii) size of market participants;
(iii) size of markets in options and in the underlying;
(iv) the extent to which prices are continuous.

Q2. Discuss briefly:
(i) what is meant by hedging;
(ii) who hedges, and why;
(iii) the most important types of hedging;
(iv) hedging in discrete time and in continuous time, and their contrasts.
(v) When, or to what extent, should an option seller hedge?

Q3. In a (Cox-Ross-Rubinstein) binomial-tree model with discount rate $1+\rho$ per period, 'up' and 'down' factors $1+u$, $1+d$ and 'up' and 'down' probabilities q, $1-q$, find the condition for q to be the risk-neutral probability.

Describe how to price an American put with strike K in an N-period binomial-tree model.

What is the connection here with the Snell envelope?

Q4. (i) If $B = (B_t)$ is Brownian motion and θ is a parameter, show that $M = (M_t)$, with

$$M_t := \exp\{\theta B_t - \frac{1}{2}\theta^2 t\},$$

is a martingale. (You may quote that the moment-generating function of $N(\mu, \sigma^2)$ is $\exp\{\mu t + \frac{1}{2}\sigma^2 t\}$.)
(ii) If Y has distribution $N(\mu, \sigma^2)$ and $X = e^Y$, X has the *log-normal distribution* $LN(\mu, \sigma^2)$. Find $E[X]$.
(iii) Show that in the Black-Scholes model, prices are log-normally distributed.
(iv) What is the relevance of (i) to the use of Girsanov's theorem when deriving the Black-Scholes formula in continuous time?

Q5. For (B_t) Brownian motion, show that
(i) (B_t^2) is a submartingale;

(ii) $(B_t^2 - t)$ is a martingale.
(iii) Find the Doob-Meyer decomposition of the submartingale (B_t^2). Hence or otherwise identify the quadratic variation of (B_t) as (t).
(iv) How is this linked with Itô calculus?
(v) How is this linked with the Black-Scholes formula?
(vi) Comment on the independent-increments assumption used in the Black-Scholes model, where Brownian motion is used as the driving-noise process.

Q6. (i) Define the Sharpe ratio.
(ii) Describe briefly, without proof, how to derive the Black-Scholes formula in continuous time from Girsanov's theorem.
(iii) Obtain a hedging strategy for the options under the Black-Scholes model in continuous time.
(iv) Comment on the limitations of this result.

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