MATH482 SOLUTIONS to EXAMINATION, 2013

Q1. (a) *Types of risk.* Institutions encounter risks of various types. Perhaps the biggest one starts at the top: how good is the board? If the board of directors, and particularly the chairman and CEO, do not have a good overview and good judgement, this alone can bring the institution down. [**3**]

Other specific types of risk include:

Market risk. This is the risk that one's current market position (the aggregate of risky assets one holds) goes down in value (things one is long on get cheaper, and/or things one is short on get dearer). [**3**]

Credit risk. This is the risk that counter-parties to one's financial transactions may default on their obligations. When this happens, debts cannot be (or are not) paid in full. Usually, payment is made in part, by negotiation between the parties (it may be cheaper to agree a partial repayment than to force the other party into bankruptcy), or by the administrators or liquidators in the case of companies. [**4**]

Operational risk. This is risk arising from the internal procedures of an institution: failure of computer systems for implementing transactions; fraudulent or unauthorised trading made possible by inadequate supervision; etc. [**4**]

Liquidity risk. This is the risk that one will be unable to implement a planned or agreed transaction because of lack of cash-in-hand to trade with, and/or willingness to trade. The Credit Crunch of 2007/8 on was caused by banks realising they had piles of toxic debt on their hands (see below), and so did not know what their balance sheets were worth; that other banks were similarly placed; hence that banks no longer trusted themselves or each other, and so refused to lend to each other. So the financial system froze up; so the real economy froze up. [**4**]

Model risk. To handle real-world phenomena of any complexity, one needs to model them mathematically. To quote Box's Dictum: All models are wrong; some models are useful. Use of an inappropriate model to set the prices at which one buys and sells exposes the institution to open-ended losses, to competitors with better models. [**4**]

(b) *Stress testing.* Financial regulators test the adequacy of the performance of a financial institution by subjecting it to *stress testing*: seeing how well its operations would perform under hypothetical but unfavourable market scenarios. This tests various aspects: their models, systems (how management and trading teams would react under pressure), capital reserves, etc. [**3**]

[Mainly seen – lectures]

Q2. (a) *Volatility.* The Black-Scholes formula involves the parameter σ (where σ^2 is the variance of the stock per unit time), called the *volatility* of the stock. In financial terms, this represents how sensitive the stock-price is to new information - how 'volatile' the market's assessment of the stock is. This volatility parameter is very important, *but* we do not know it; instead, we have to *estimate* the volatility for ourselves. There are two approaches: [**3**]
(b) *Historic volatility*: here we use Time Series methods to estimate σ from past price data. Clearly the more variability we observe in runs of past prices, the more volatile the stock price is, and given enough data we can estimate σ in this way. [**4**]
Implied volatility: match observed option prices to theoretical option prices. For, the price we see options traded at tells us what the *market* thinks the volatility is (estimating volatility this way works because the dependence is monotone). [**4**]
Volatility surface. If the Black-Scholes model were perfect, these two estimates would agree (to within sampling error). But discrepancies can be observed, which shows the imperfections of our model. Volatility graphed against price S, or strike K, typically shows a *volatility smile* (or even smirk). Graphed against S and K in 3 dimensions, we get the *volatility surface.* [**4**]
(c) Volatility dependence is given by *vega* $:= \partial c/\partial\sigma$ for calls, $\partial p/\partial\sigma$ for puts. From the Black-Scholes formula (which gives the price explicitly as a function of σ), one can check by calculus that $\partial c/\partial\sigma > 0$, and similarly for puts (or, use the result for calls and put-call parity). *Options like volatility.* The more uncertain things are (the higher the volatility), the more valuable protection against adversity becomes (the higher the option price). [**3**]
(d) The classical view of volatility is that it is caused by future uncertainty, and shows the market's reaction to the stream of new information. However, studies taking into account periods when the markets are open and closed [there are only about 250 trading days in the year] have shown that the volatility is less when markets are closed than when they are open. This suggests that *trading itself is one of the main causes of volatility.* [**4**]

The introduction of a small transaction tax would have the effect of decreasing trading. This would increase market stability: trading is one of the causes of volatility; options like volatility. So trading tends to cause an increase in trading in options, and so on. Ultimately this tends to induce market instability. So conversely, market stability would benefit from a reduction in trading volumes caused by a transaction tax. [**3**]
[Mainly seen – lectures]

Q3. *Discrete and continuous Black-Scholes models and formulae.*
(a) In the *discrete* Black-Scholes (BS) model, we use the (Cox-Ross-Rubinstein, CRR) *binomial tree* model of 1979. At each step, the price can go up or down; we use a 'recombining' tree, so that 'up' and 'down' paths link. The price at expiry thus depends on the number of up and down steps, and this has a *binomial distribution.* We can calculate the payoff at each terminal node. The tree has a unique *risk-neutral measure*, P^* say, under which discounted asset prices become martingales. The price of the option at any time is thus the P^* expectation of the discounted value of the payoff, and we can find this as a suitable *binomial sum.* This gives the *discrete BS formula.* [**4**]

Now just as the binomial distribution has a histogram approximating a suitable normal density, the binomial sum in the discrete BS formula also has a limit, given by two terms, both involving Φ, one involving the stock price S, the other the strike price K. This limit of the discrete BS formula is the (continuous) *Black-Scholes formula* of 1973. [**4**]

Not only does the *formula* have a limit, as above, the *model* has a limit. Brownian motion is a suitable limit of random walks. So we can treat the continuous BS formula directly via BM (as Black and Scholes did in 1973), or indirectly via the CRR tree of 1979. [**3**]

(b) When we pass from the real probability measure P to the risk-neutral probability measure (or equivalent martingale measure, EMM) P^*, the *mean* return μ on the stock is lost, and replaced by the riskless return rate r. What survives is the relevant *variance*, or rather its square root, the *volatility*, σ. So: both discrete and continuous BS formulae involve σ but *not* μ. [**4**]

(c) For an American put, we have at each node of the tree the option of exercising early. We calculate both the option value and the optimal exercise strategy by working backwards through the tree:

1. Draw the tree, and fill in the stock price at each node.
2. Using the strike price K and the prices at the *terminal nodes*, fill in the payoffs ($f_{N,j} = \max[K - Su^j d^{N-j}, 0]$) at the terminal nodes.
3. Go back one time-step. Fill in the 'European' value at the penultimate nodes as the discounted values of the upper and lower right (terminal) values, under P^* - '$p^* \times$ lower right plus $1 - p^* \times$ upper right'. Fill in the 'intrinsic' (or early-exercise) value. The American put value is the higher of these.
4. Iterate, working back down the tree to the root. The value of the American put at time 0 is the value at the root. The nodes split into the 'early-exercise region' and the 'continuation region'. [**10**]

[Seen, lectures]

Q4. *Martingale transforms; stochastic integrals; trading and gains from trade.*
(a) Call a process $C = (C_n)_{n=1}^{\infty}$ *previsible* (or *predictable*) if

$$C_n \text{ is } \mathcal{F}_{n-1} - \text{measurable for all } n \geq 1. \qquad [\mathbf{2}]$$

(b) Think of C_n as your stake on play n (C_0 is not defined, as there is no play at time 0). Previsibility says that you have to decide how much to stake on play n based on the history *before* time n (i.e., up to and including play $n-1$). Your winnings on game n are $C_n \Delta X_n = C_n(X_n - X_{n-1})$. Your total (net) winnings up to time n are

$$Y_n = \sum_{k=1}^{n} C_k \Delta X_k = \sum_{k=1}^{n} C_k (X_k - X_{k-1}).$$

We write

$$Y = C \bullet X, \qquad Y_n = (C \bullet X)_n, \qquad \Delta Y_n = C_n \Delta X_n$$

($(C \bullet X)_0 = 0$ as $\sum_1^0$ is empty), and call $C \bullet X$ the *martingale transform* of X by C. [**4**]

(c) **Theorem**. (i) If C is bounded and previsible and X is a martingale, $C \bullet X$ is a martingale null at zero.

Proof. With $Y = C \bullet X$ as above,

$$\begin{aligned} E[Y_n - Y_{n-1} | \mathcal{F}_{n-1}] &= E[C_n (X_n - X_{n-1}) | \mathcal{F}_{n-1}] \\ &= C_n E[(X_n - X_{n-1}) | \mathcal{F}_{n-1}] \end{aligned}$$

(as C_n is bounded, so integrable, and $\mathcal{F}_{n-1}$-measurable, so can be taken out),

$$= 0, \qquad \text{(as } X \text{ is a martingale).} \qquad [\mathbf{10}]$$

(d) In mathematical finance, X plays the role of a price process, C plays the role of our trading strategy, and the mg transform $C \bullet X$ plays the role of our gains (or losses!) from trading. The *previsibility* of C corresponds to *no insider trading*: one has to decide on one's current trades in the light of current information, not future information. [**4**]
(e) Similarly in continuous time, where $C \bullet X$ becomes the stochastic integral $\int_0^t C(s) dX(s)$. Previsibility here means $C(t) \in \mathcal{F}_{t-} := \bigcup_{s<t} \mathcal{F}_s$: one has to decide on one's current trades at time t "just before" t – in ignorance of any new price-sensitive information at t. [**5**]
[Seen – lectures]

Q5. *Geometric Brownian Motion (GBM).* (a) Consider the Black-Scholes model, with dynamics given by the stochastic differential equation (SDE)

$$dB_t = rB_t dt, \qquad dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{GBM}$$

The interpretation here is that B_t is our bank account at time t – money invested risklessly at rate r, so growing exponentially. The risky stock S has a similar term, this time with growth-rate μ (which models the systematic part of the price dynamics), plus a second term which models the risky part. The uncertainty in the economic and financial climate is represented by the Brownian motion (BM) $W = (W_t)$; this is coupled to the stock-price dynamics via the paramater σ, the *volatility*, which measures how sensitive this particular risky stock is to changes in the overall economic climate. [**7**]
(b) Discounting the prices by e^{rt}, the discounted asset prices $\tilde{S}_t := e^{-rt} S_t$ have dynamics given, as before, by

$$\begin{aligned} d\tilde{S}_t &= -re^{-rt} S_t dt + e^{-rt} dS_t \\ &= -r\tilde{S}_t dt + \mu \tilde{S}_t dt + \sigma \tilde{S}_t dW_t \\ &= (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dW_t. \end{aligned}$$

Thus discounting changes the rate μ on the RHS of (GBM) to $\mu - r$. [**7**]
(c) Now use Girsanov's Theorem to change from the real probability measure P to an equivalent probability measure P^* under which the μdt in (GBM) is rdt. Then under P^*, the stock-price dynamics become

$$d\tilde{S}_t = \sigma \tilde{S}_t dW_t \qquad \text{(under } P^*\text{)}.$$

Integrating, $\tilde{S}$ on the left is a stochastic integral w.r.t. Brownian motion – which is a martingale. This P^* is the *equivalent martingale measure (EMM)*, or *risk-neutral measure*. The EMM is that in the continuous-time version of the Fundamental Theorem of Asset Pricing: *to price assets, take expectations of discounted prices under the risk-neutral measure.* This leads to the Black-Scholes formula by direct probabilistic means, rather than via the Black-Scholes PDE. [**7**]
(d) In the Black-Scholes model, markets are complete. So the EMM is unique. This is a result of the representation theorem for Brownian martingales: *any* Brownian martingale can be represented as a stochastic integral w.r.t. BM. Completeness results from the *continuity* of the paths of BM. [**4**]
[Mainly seen – lectures]

Q6. *Real options.* (a) With starting value x, to solve the optimal stopping problem

$$V(x) := \max_{\tau} E[(X_\tau - I)e^{-r\tau}]$$

(discounting as usual). This gives the best discounted profit, buying an asset of value X for a cost I, at time τ chosen optimally. [**4**]
(b) If $\mu \leq 0$, the (mean) value of the project will decrease. So we invest immediately if $x > I$ (with immediate profit $x - I > 0$), and do not invest otherwise. If $\mu > r$, the (mean) growth will swamp the riskless interest rate, so the investment is worthwhile, and we should again invest immediately as there is no point in waiting. If $\mu = r$, there is no point in taking the risk of investing, so we should not invest. [**4**]
(c) There remains the case $0 < \mu < r$. Using the infinitesimal generator, one gets the differential equation (Bellman equation)

$$\frac{1}{2}\sigma^2 x^2 V''(x) + \mu x V'(x) - rV(x) = 0,$$

with $V(0) = 0$ (we get nothing from something worth nothing). A suitable trial solution is $V(x) = Cx^p$. This leads to a quadratic equation in p:

$$Q(p) := \frac{1}{2}\sigma^2 p(p-1) + \mu p - r = 0. \qquad [\mathbf{6}]$$

The product of the roots is negative, and $Q(0) = -r < 0$, $Q(1) = \mu - r < 0$. So one root $p_1 > 1$ and the other $p_2 < 0$.
(d) The general solution is $V(x) = C_1 x^{p_1} + C_2 x^{p_2}$, but from $V(0) = 0$ we get $C_2 = 0$, so $V(x) = C_1 x^{p_1}$, or $V(x) = Cx^{p_1}$. If x^* is the critical value at which it is optimal to invest, 'value matching' and 'smooth pasting' give

$$V(x^*) = x^* - I, \qquad V'(x^*) = 1. \qquad [\mathbf{6}]$$

From these two equations, we can find C and x^*:

$$V'(x^*) = Cp_1(x^*)^{p_1 - 1} = 1, \qquad C = (x^*)^{1-p_1}/p_1.$$

Then value matching gives

$$C(x^*)^{p_1} = x^* - I, \qquad x^*/p_1 = x^* - I, \qquad x^* = \frac{p_1}{(p_1 - 1)} I.$$

So we should not invest if the initial value x is below $x^* = qI$, where $q := p_1/(p_1 - 1)$ ("Tobin's q").

Note that this is different from the traditional NPV (net present value) accountancy approach. [**5**]
[Seen, lectures]