MATH482 EXAMINATION SOLUTIONS, 2014

Q1. (i) *Arbitrage.* An arbitrage opportunity is the possibility of making a riskless profit – a trading strategy in which one starts with nothing, never makes a loss, but might make a profit.
(ii) The assumption of absence of arbitrage is unrealistic – but no more so than the other assumptions of a perfect market (same interest rate for borrowing and lending, no liquidity restrictions/transaction costs/taxes etc.).

Small arbitrage opportunities may be present and persist. Large ones would attract the attention of speculators and other market participants. This would change the balance of supply and demand, and so prices, so the arbitrage opportunity would shrink (be 'arbitraged away').
(iii) If the market is arbitrage-free,
(a) there exists an equivalent martingale measure;
(b) equilibrium may be possible (it is impossible if arbitrage is present). **[7]**
(ii) *Completeness.* A market is *complete* if every contingent claim can be replicated, by a combination of stock and cash. Now any such combination may be priced uniquely (count the cash; count the stock; look up the stock price; do the arithmetic). So in a complete market, prices are unique, and do not depend on the attitude to risk of investors (their utility function).

In reality, prices are *not* unique – typically, they fill out an interval, the *bid-ask spread.* This reflects the difference between buying and selling, and the need to have a margin between the two to cover overheads etc. **[6]**
(iii) *Equivalent martingale measures (EMMs).* We model the uncertainty in risky stocks by a probability measure, P say. As stock prices occur in the real economy, we call P the real(-world) measure. As always, we *discount* prices over time, to work with real prices rather than nominal prices. Two measures P and Q are called *equivalent* if they have the same null sets (sets of probability 0) – i.e., the same things are possible/impossible under both P and Q. We call P^* an/the *equivalent martingale measure* if under P^*, *discounted asset prices become martingales.* The two key results re:
No-arbitrage Theorem. The market has *no arbitrage* (is *NA*) iff EMMs *exist.*
Completeness Theorem. The market is *complete* iff EMMs are *unique.* **[6]**
(iv) *Risk-neutral valuation.* For complete NA markets, the unique EMM P^* is called the *risk-neutral measure.* The *risk-neutral valuation formula* says that for a complete NA market, asset prices at time t can be calculated as the conditional P^*-expectation of the discounted payoff at expiry T. **[6]**
[Seen – lectures + Mock Exam 2013]

Q2. *Put-Call Parity.* The price (or value) of the portfolio at time t is $Ke^{-r(T-t)}$, that is,

$$S + P - C = Ke^{-r(T-t)}. \qquad (PCP) \quad [\mathbf{5}]$$

Proof. We prove this by arbitrage. Consider a portfolio which is long one asset, long one put and short one call; write Π (or Π_t) for its value. So

$$\Pi = S + P - C \qquad \text{(S: long asset; P: long put; -C: short call).}$$

Recall that the payoffs at expiry are:

$$\begin{cases} \max(S-K,0) & \text{or} \quad (S-K)_+ \quad \text{for a call,} \\ \max(K-S,0) & \text{or} \quad (K-S)_+ \quad \text{for a put.} \end{cases}$$

So the value of the above portfolio at expiry is K:

$$\begin{cases} S + 0 - (S-K) = K & \text{if } S \geq K \\ S + (K-S) - 0 = K & \text{if } K \geq S, \end{cases}$$

This portfolio thus guarantees a payoff K at time T (and so is financially equivalent to cash K, a riskless asset). **[8]**

This riskless equivalent is worth $Ke^{-r(T-t)}$ at time t. So directly from this, so too is the portfolio, proving (PCP).

Alternatively, one can give an explicit arbitrage argument. If the portfolio is offered for sale at time t too cheaply – at a price $\Pi < Ke^{-r(T-t)}$ – *buy* it, *borrow* $Ke^{-r(T-t)}$ from the bank, and pocket a positive profit $Ke^{-r(T-t)} - \Pi > 0$. At T the portfolio yields K (above), while the bank debt has grown to K. Clear the cash account – use the one to pay off the other – thus locking in the earlier profit, which is *riskless*. If on the other hand the portfolio is offered for sale at time t at too high a price – at price $\Pi > Ke^{-r(T-t)}$ – do the exact opposite. *Sell the portfolio short* – that is, *buy its negative*, long one call, short one put, short one asset, for $-\Pi$, and *invest* $Ke^{-r(T-t)}$ in the bank, pocketing a positive profit $-(-\Pi) - Ke^{-r(T-t)} = \Pi - Ke^{-r(T-t)} > 0$. At time T, the bank deposit has grown to K. Clear the cash account – use this to meet the obligation K on the portfolio sold short, again locking in the earlier riskless profit.

Thus the rational price for the portfolio at time t is *exactly* $Ke^{-r(T-t)}$. *Any other price* presents arbitrageurs with an arbitrage opportunity (to make and lock in a riskless profit) – which they will take! This proves (PCP). **[8]**

Interpretation. The value of the portfolio $S + P - C$ is the *discounted value of the riskless equivalent.* **[4]**

[Seen – lectures + Mock Exam 2013]

Q3 *Doubling strategy.* (i) With N the number of losses before the first win:

$$P(N = k) = P(L, L, \cdots, L(k \text{ times}), W) = (\frac{1}{2})^k . \frac{1}{2} = (\frac{1}{2})^{k+1}.$$

That is, N is geometrically distributed with parameter 1/2. As

$$\sum_{k=0}^{\infty} P(N = k) = \sum_0^{\infty} (\frac{1}{2})^{k+1} = \frac{1}{2}/(1 - \frac{1}{2}) = 1,$$

$P(N < \infty) = 1$: $N < \infty$ a.s. So one is certain to win eventually. **[5]**

(ii) Let S_n be one's fortune at time n. When $N = k$, one has losses at trials $1, 2, 3, \ldots, k$, with losses $1, 2, 4, \ldots, 2^{k-1}$, followed by a win at trial $k + 1$ (of 2^k). So one's fortune then is

$$2^k - (1 + 2 + 2^2 + \ldots + 2^{k-1}) = 2^k - (2^k - 1) = 1,$$

summing the finite geometric progression. So one's eventual fortune is $+1$ (which, by (i), one is certain to win eventually). **[5]**

(iii) N has PGF

$$P(s) := E[s^N] = \sum_{n=0}^{\infty} s^k P(N = k) = \sum_0^{\infty} s^k . (\frac{1}{2})^{k+1}$$

$$= \frac{1}{2} \sum_0^{\infty} (\frac{1}{2} s)^k = \frac{1}{2}/(1 - \frac{1}{2} s) = 1/(2 - s) :$$

$$P'(s) = E[N s^{N-1}] = (2 - s)^{-2}; \qquad P'(1) = E[N] = 1.$$

So the mean number of losses is 1, and the mean time the game lasts is 2. **[5]**

(iv) As with the simple random walk: this is an impossible strategy to use in reality, for two reasons:

(a) It depends on one's opponent's cooperation. What is to stop him trying this on you? If he does, the game degenerates into a simple coin toss, with the winner walking away with a profit of 1 (pound, or million pounds, say) – suicidally risky. **[5]**

(b) Even with a cooperative opponent, it relies on the gambler having an unlimited amount of cash to bet with, or an unlimited line of credit – both hopelessly unrealistic in practice. **[5]**

[Seen – Problems]

Q4 *Properties of conditional expectation.*

The *conditional expectation* of a random variable Y with $E[|Y|] < \infty$ *given* a σ-field $\mathcal{C}$, $E[Y|\mathcal{C}]$, is defined by:

$E[Y|\mathcal{C}]$ is $\mathcal{C}$-measurable; **[3]**

$$\int_C E[Y|\mathcal{C}]dP = \int_C Y dP \qquad \forall C \in \mathcal{C} \quad a.s. \qquad [3]$$

If $\mathcal{C}$ is the trivial σ-field $\{\emptyset, \Omega\}$, $E[Y|\mathcal{C}] = E[Y]$: a conditional expectation given no information is just an ordinary unconditional expectation. **[2]**

If $\mathcal{C} = \mathcal{F}$ is the whole σ-field (in the definition of the probability space), $E[Y|\mathcal{C}] = Y$: a conditional expectation given full information is the random variable itself (no randomness left to average over). **[2]**

Iterated conditional expectation (i). If $\mathcal{C} \subset \mathcal{B}$, $E[E(Y|\mathcal{B})|\mathcal{C}] = E[Y|\mathcal{C}] \quad a.s.$

Proof. $E_{\mathcal{C}} E_{\mathcal{B}} Y$ is $\mathcal{C}$-measurable, and for $C \in \mathcal{C} \subset \mathcal{B}$,

$$\int_C E_{\mathcal{C}}[E_{\mathcal{B}} Y] dP = \int_C E_{\mathcal{B}} Y dP \qquad (\text{definition of } E_{\mathcal{C}} \text{ as } C \in \mathcal{C})$$

$$= \int_C Y dP \qquad (\text{definition of } E_{\mathcal{B}} \text{ as } C \in \mathcal{B}).$$

So $E_{\mathcal{C}}[E_{\mathcal{B}} Y]$ satisfies the defining relation for $E_{\mathcal{C}} Y$. Being also $\mathcal{C}$-measurable, it *is* $E_{\mathcal{C}} Y$ (a.s.). // **[6]**

Iterated conditional expectation (ii). If $\mathcal{C} \subset \mathcal{B}$, $E[E(Y|\mathcal{C})|\mathcal{B}] = E[Y|\mathcal{C}] \quad a.s.$

Proof. $E[Y|\mathcal{C}]$ is $\mathcal{C}$-measurable, so $\mathcal{B}$-measurable as $\mathcal{C} \subset \mathcal{B}$, so $E[.|\mathcal{B}]$ has no effect on it ('taking out what is known': given $\mathcal{B}$, we know Y, so it counts as a constant and we can take it out through integrals, i.e. expectations). **[4]**

Conditional Mean Formula. $E[E(Y|\mathcal{B})] = EY \quad P - a.s.$

Proof. Take $\mathcal{C} = \{\emptyset, \Omega\}$. // **[2]**

Projections. Above, take $\mathcal{B} = \mathcal{C}$:

$$E[E[X|\mathcal{C}]|\mathcal{C}] = E[X|\mathcal{C}].$$

This says that the operation of taking conditional expectation given a sub-σ-field $\mathcal{C}$ is *idempotent* – doing it twice is the same as doing it once. Also, taking conditional expectation is a *linear* operation (it is defined via an integral, and integration is linear). So as in Linear Algebra, being idempotent and linear it is called a *projection* (Example: $(x, y, z) \mapsto (x, y, 0)$ projects from 3-dimensional space onto the (x, y)-plane). **[3]**

[Seen – lectures]

Q5. *Vega for calls.* With

$$\phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}, \qquad \Phi(x) := \int_{-\infty}^{x} \phi(u)du$$

the standard normal density and distribution functions, $\tau := T - t$ the time to expiry, the Black-Scholes call price is

$$C_t := S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \qquad (BS)$$

$$d_1 := \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 := \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau}.$$

(i) So

$$\phi(d_2) = \phi(d_1 - \sigma\sqrt{\tau}) = \frac{e^{-\frac{1}{2}(d_1 - \sigma\sqrt{\tau})^2}}{\sqrt{2\pi}} = \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}}.e^{d_1\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau} :$$

$$\phi(d_2) = \phi(d_1).e^{d_1\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau}. \qquad [\mathbf{5}]$$

(ii) Exponentiating the definition of d_1,

$$e^{d_1\sigma\sqrt{\tau}} = (S/K).e^{r\tau}.e^{\frac{1}{2}\sigma^2\tau}. \qquad [\mathbf{3}]$$

(iii) Combining (i) and (ii),

$$\phi(d_2) = \phi(d_1).(S/K).e^{r\tau} : \qquad Ke^{-r\tau}\phi(d_2) = S\phi(d_1). \qquad (*) \quad [\mathbf{5}]$$

Differentiating (BS) partially w.r.t. σ gives

$$v := \partial C/\partial\sigma = S\phi(d_1)\partial d_1/\partial\sigma - Ke^{-r\tau}\phi(d_2)\partial d_2/\partial\sigma.$$

This and $(*)$ give

$$v := \partial C/\partial\sigma = S\phi(d_1)\partial(d_1 - d_2)/\partial\sigma = S\phi(d_1)\partial\sigma\sqrt{\tau})/\partial\sigma = S\phi(d_1)\sqrt{\tau} > 0.$$

[5]

Vega for puts.

The same argument gives $v := \partial P/\partial\sigma > 0$, starting with the Black-Scholes formula for puts. Equivalently, we can use put-call parity

$$S + P - C = Ke^{-r\tau} : \qquad \partial P/\partial\sigma = \partial C/\partial\sigma > 0. \qquad [\mathbf{4}]$$

Interpretation: "Options like volatility": the higher the volatility, i.e. the more uncertainty there is, the more the "insurance policy" of an option is worth. **[3]**

[Seen – Problems]

Q6 *Black-Scholes formula.* (i) There is no point in investing in a risky stock, at mean return rate μ, if one can do as well or better by investing risklessly at rate r. So one should invest all one's funds in cash, unless $\mu > r$. The excess return $\mu - r$ is risky, and σ measures the risk involved. The *Sharpe ratio* is $\lambda := (\mu - r)/\sigma$. This is the usual measure to use, e.g. in deciding between one risky investment and another. **[3]**

By Markowitzian diversification, the manager would wish to have some cash and some stock; he would increase the proportion of his funds held in stock as λ increases. **[2]**

(ii) The Black-Scholes formula gives the value of a European option on a risky stock with dynamics as in $(*)$. One should:

(a) pass from the real-world (or physical) probability measure P to the *risk-neutral* probability measure P^* – the probability measure equivalent to P (same events possible, same events impossible), but under which the dynamics are

$$dS_t = S_t(rdt + \sigma dB_t), \qquad [4]$$

– i.e., one replaces μ by r;

(b) *discount* (by the riskless interest rate r), so passing from nominal prices to real prices. This replaces the dynamics by

$$dS_t = S_t.\sigma dB_t. \qquad [4]$$

This can be integrated (stochastic exponential), to give S_T. The Risk-Neutral Valuation Formula gives the option price as the expectation of the payoff (a simple function of S, $(S-K)_+$ or $(K-S)_+$) under P^*. The resulting integral can be evaluated in two terms, both involving Φ, one involving also S_T, the other the discounted strike price K. **[4]**

(iii) The Black-Scholes formula does not involve μ, as it is replaced by r in step (a) above (technically, this is an application of *Girsanov's theorem*). **[3]**

(iv) Here σ is the *volatility* of the stock, a measure of how changeable it is as market conditions change. We do not know it, so have to estimate it. Since the Black-Scholes price is an increasing function of σ ('options like volatility'), one can infer σ (or what the market thinks it is) by matching it to the value giving the price at which the relevant option is currently trading (*implied volatility*). **[3]**

Alternatively, one can look at the price process over time and use Time Series methods from Statistics to estimate σ (*historic volatility*). **[2]**

[Seen – lectures] N. H. Bingham