Math482 EXAMINATION SOLUTIONS 2015

Q1. *Perfect Markets.*

For simplicity, we shall confine ourselves to option pricing in the simplest (idealised) case, of a *perfect*, or *frictionless*, market. This entails various assumptions:

a. *No transaction costs.* We assume that there is no financial friction in the form of transaction costs (one can include transaction costs in the theory, but this is considerably harder). **[3]**

b. *No taxes.* We assume similarly that there are no taxes. We note that a *Tobin tax*, designed partly to damp down excessive volumes of trading and partly to raise money for good causes, has recently been suggested. **[3]**

c. *Same interest rates for borrowing and for lending.* This is clearly unrealistic, as banks make their money on the difference. But it is a reasonable first approximation, and simplifies such things as arbitrage arguments. **[3]**

d. *Perfect information.* We assume that all market participants have perfect information about the past history of price movements, but have no foreknowledge of price-sensitive information (i.e. no insider trading) – and no information asymmetry: all participants are equally knowledgeable. **[3]**

e. *No liquidity restrictions.* That is, one can buy or sell unlimited quantities of stock at the currently quoted price. However, in a crisis, no-one wants to trade, and liquidity dries up. **[3]**

f. *Economic agents are price takers and not price makers.* In practice, this is true for small market participants but not for large ones. Big trades do move markets (price is the level at which supply and demand balance; big trades affect this balance significantly). This restriction emphasizes the difference between Economics and Finance. Much of Economics is concerned with *how prices are arrived at* (supply and demand, etc.). In Finance, at least here, we take prices as given. **[3]**

g. *No credit risk.* Perfect markets assume that all market participants are willing and able to honour their commitments. This ignores the risk of bankruptcy, etc. **[3]**

h.*No restriction on order size; no delay in executing orders.* In practice, executing small orders is uneconomic, so there are size limitations. Also, orders are dealt with in job lots, for efficiency. Delays do occur in executing orders, particularly large ones. **[2]**

Other risks, e.g.: fraud; human error; insider trading; etc. **[2]**

[Seen – lectures]

Q2. (i) A market has *no arbitrage* (is *NA*, or is *viable*) if there is no trading strategy that starts from value zero, always has final value non-negative and has positive final value with positive probability – that is, if one cannot create riskless profit ('free money'), even some of the time. **[2]**

A market is *complete* if every contingent claim can be replicated by a suitable combination of cash and the underlying – that is, every option has a hedging portfolio replicating it as a combination of cash and stock. **[2]**

An *equivalent martingale measure (EMM)* (or *risk-neutral measure*) is a measure P^* (or Q) equivalent to the real (or physical), measure P (in the probability space of our model for the market) (= has the same null sets as P) under which *discounted asset prices are martingales* (P^*-martingales). **[2]**

(ii) The *No-Arbitrage Theorem* says that the model is NA iff EMMs *exist*. **[2]**

The *Completeness Theorem* says that a NA market is complete iff EMMs are *unique*. **[2]**

The *Fundamental Theorem of Asset Pricing (FTAP)* combines these: a market is NA and complete iff EMMs exist and are unique. **[2]**

The *Risk-Neutral Valuation Formula (RNVF)* quantifies this by saying that the price at time $t \in [0, T]$ of an asset with payoff h at expiry T is

$$V_n(H) = (1+r)^{-(N-n)} E^*[V_N(H)|\mathcal{F}_n] = (1+r)^{-(N-n)} E^*[h|\mathcal{F}_n]$$

(H is the *hedging strategy* above, r the riskless interest rate). **[2]**

(iii) Real markets do show arbitrage opportunities. These are given by traders who buy or sell at incorrect prices. These can be identified and exploited by arbitrageurs, who buy what is sold too cheaply and sell what is bought too dearly, pocket the (riskless) profit, sell or buy at the correct price – and then repeat, until the trader thus offering 'free money' is driven from the market, solvent or otherwise. So arbitrage opportunities are transient, and are a 'second-order effect' – are of secondary importance overall. **[6]**

(iv) Real markets are incomplete. This is observed in non-uniqueness of real prices – the *bid-ask spread* ('You'd better shop around'). It is also shown in the impossibility of option sellers (say) hedging their position perfectly. But they do not need to. Institutions sell options not in a defensive way, seeking to cover themselves completely against all possible loss. They provide a service in offering option buyers *insurance* against adverse price movements, and make most of their money on this. They will be content to be able to hedge against most, not all, of their possible losses, on options expiring in the money and exercised against them. **[5]**

[Seen – lectures]

Q3 (*Rubber options*). The price of rubber now is \$ 185/100kg. Next year, it will be 195 or 180, each with positive probability. The strike is $K = 185$.
Risk-neutral measure. We determine p^*, the 'up probability', so as to make the price a martingale. Neglecting interest, this gives

$$185 = p^*.195 + (1 - p^*).180 = 180 + 15p^*, \qquad 5 = 15p^*, \qquad p^* = 1/3.$$

(i) *Pricing.* There is no discounting, so the value V_0 at time 0 is the P^*-expectation E^* of the payoff H next year:

$$V_0 = E^*[H] = p^*.10 + (1 - p^*).0 = 10p^* = 10.1/3 = 3.33. \qquad \textbf{[5]}$$

(ii) *Hedging.* The call C is financially equivalent to a portfolio Π consisting of a combination of cash and rubber, as the binomial model is *complete* – all contingent claims (options etc.) can be *replicated.* To find *which* combination (ϕ_0, ϕ_1) of cash and rubber, we solve two simultaneous linear equations:

$$\begin{aligned} \text{Up}: &\quad 10 = \phi_0 + 195\phi_1, \\ \text{Down}: &\quad 0 = \phi_0 + 180\phi_1. \end{aligned}$$

Subtract: $10 = 15\phi_1$: $\phi_1 = 2/3$.
Substitute: $\phi_0 = -180\phi_1 = -180 \times 2/3 = -120$.
So C is equivalent to the portfolio $\Pi = (-120, 2/3)$: *long*, 2/3 of 100kg rubber, *short*, \$ 120 cash.
Check: in a year's time,
Rubber up: Π is worth (2/3).195 - 120 = 130 - 120 = 10, as H is;
Rubber down: Π is worth (2/3).180 - 120 = 0, as H is. **[5]**
Arbitrage. By (i) and (ii), you know C and Π are worth 3.33 now.
(iii) If you see C being traded (= bought and sold) for *more* than it is worth, *sell* it, for 4. You can buy it, or equivalently the hedging portfolio Π, for 3.33. Pocket the risk-free profit (67 cents per option) now. The hedge enables you to meet your obligations to the option holder, at zero net cost. **[4]**
(iv) If C is being traded for *less* than it is worth, *buy* it, for 3. You can sell it, equivalently Π, for 3.33. Pocket the risk-free profit (33 cents) now. Again, the option payoff clears your position, at zero net cost. **[4]**
(v) Call options of rubber are bought by manufacturers of tyres, etc., as an insurance policy against prices moving up (e.g., after a bad harvest). **[3]**
(vi) Put options of rubber are bought by growers, as an insurance policy against prices moving down (e.g., after a good harvest). **[3]**
Options of either kind may be traded opportunistically, by speculators. **[1]**
[Similar seen: lectures and problems]

Q4. (i) *Brownian motion (BM)* $W = (W(t))$, or $W = (W_t)$ (whichever is more convenient) is defined as the process with:
(a) $W(0) = 0$; **[1]**
(b) W has stationary independent Gaussian increments, with
$W(s+t) - W(s) \sim N(0,t)$ for all $s \geq 0$; **[2]**
(c) the paths $t \mapsto W(t)$ are continuous (in t, a.s. in ω). **[2]**
(ii) *Occurrence of BM.* Brownian motion is widely used to model *driving noise* – the noise, or error, or unpredictability, in the world, that makes the future unpredictable. It is the signature of phenomena which are the visible end-product of *a large number of individually negligible causes.* Mathematically, this is a result of the *Central Limit Theorem (CLT).* In finance, one sees this clearly in BM as the limit of the binomial tree model. **[5]**
(iii) *Brownian covariance.* For $s \leq t$,

$$W_t = W_s + (W_t - W_s), \qquad W_s W_t = W_s^2 + W_s(W_t - W_s).$$

Take expectations: on the left we get $cov(W_s, W_t)$. The first term on the right is, as $E[W_s] = 0$, $var(W_s) = s$. As Brownian motion (BM) has independent increments, $W_t - W_s$ is independent of W_s, so

$$E[W_s(W_t - W_s)] = E[W_s].E[W_t - W_s] = 0.0 = 0.$$

Combining, $cov(W_s, W_t) = s$ for $s \leq t$. Similarly, for $t \leq s$ we get t. Combining, $cov(W_s, W_t) = \min(s,t)$. **[5]**
(iv) *Brownian scaling.* With $W_c(t) := W(c^2 t)/c$,

$$cov(W_c(s), W_c(t)) = E[W(c^2 s)/c.W(c^2 t)/c] = c^{-2} \min(c^2 s, c^2 t) = \min(s,t) = cov(W_s, W_t).$$

So W_c has the same mean 0 and covariance $\min(s,t)$ as BM W. It is also (from its definition) continuous, Gaussian, stationary independent increments etc. So it has all the defining properties of BM. So it *is* BM. **[5]**
(v) *Scale in finance.* Small agents are price takers rather than price makers – they can trade as much as they want at their own small scale, without shifting prices. By contrast, large agents are price makers rather than price takers: large trades shift prices (as they affect the current balance of supply and demand). Use of BM ignores this difference, because of Brownian scaling (iv), and this is a limitation of the Black-Scholes model. **[5]**
[Mainly seen – lectures and problems]

Q5 *Ornstein-Uhlenbeck (OU) process.*
(i) The OU SDE $dV = -\beta V dt + \sigma dW$ (OU) models the velocity of a diffusing particle. The $-\beta V dt$ term is *frictional drag*; the σdW term is *noise*. **[3]**
(ii) $e^{-\beta t}$ solves the corresponding homogeneous DE $dV = -\beta V dt$. So by variation of parameters, take a trial solution $V = Ce^{-\beta t}$. Then

$$dV = -\beta Ce^{-\beta t}dt + e^{-\beta t}dC = -\beta V dt + e^{-\beta t}dC,$$

so V is a solution of (OU) if $e^{-\beta t}dC = \sigma dW$, $dC = \sigma e^{\beta t}dW$, $C = c + \sigma \int_0^t e^{\beta u}dW$. So with initial velocity v_0, $V = e^{-\beta t}C$ is

$$V = v_0 e^{-\beta t} + \sigma e^{-\beta t}\int_0^t e^{\beta u}dW_u. \qquad \textbf{[5]}$$

(iii) V comes from W, Gaussian, by linear operations, so is Gaussian.
V_t has mean $v_0 e^{-\beta t}$, as $E[e^{\beta u}dW_u] = \int_0^t e^{\beta u}E[dW_u] = 0$.
By the Itô isometry, V_t has variance

$$E[(\sigma e^{-\beta t}\int_0^t e^{\beta u}dW_u)^2] = \sigma^2 e^{-2\beta t}\int_0^t (e^{\beta u})^2 du$$

$$= \sigma^2 e^{-2\beta t}[e^{2\beta t} - 1]/(2\beta) = \sigma^2[1 - e^{-2\beta t}]/(2\beta).$$

So V_t has distribution $N(v_0 e^{-\beta t}, \sigma^2(1 - e^{-2\beta t})/(2\beta))$. **[5]**
(So the limit distribution as $t \to \infty$ is $N(0, \sigma^2/(2\beta))$, the *Maxwell-Boltzmann distribution* of Statistical Mechanics.)
(iv) For $u \geq 0$, the covariance is $cov(V_t, V_{t+u})$, which is

$$\sigma^2 E[e^{-\beta t}\int_0^t e^{\beta v}dW_v . e^{-\beta(t+u)}(\int_0^t + \int_t^{t+u})e^{\beta w}dW_w].$$

By independence of Brownian increments, $\int_t^{t+u}$ contributes 0, so by above

$$cov(V_t, V_{t+u}) = e^{-\beta u}var(V_t) = \sigma^2 e^{-\beta u}[1 - e^{-2\beta t}]/(2\beta) \to \sigma^2 e^{-\beta u}/(2\beta) \quad (t \to \infty).$$

[5]
(v) V is Markov (a diffusion), being the solution of the SDE (OU). **[3]**
(vi) The process shows *mean reversion* – a strong push towards the central value. This is characteristic of interest rates (under normal conditions). The financial relevance is to the *Vasicek model* of interest-rate theory. **[4]**
[Seen, lectures.]

Q6. *American options – infinite horizon.*

We deal with a *put* option – giving the right to sell at the strike price K, at any time τ of our choosing. This τ has to be a *stopping time*: we decide whether or not to stop at τ based on information already available.

Under the risk-neutral measure, the SDE for GBM becomes

$$dX_t = rX_t dt + \sigma X_t dB_t. \qquad (GBM_r) \ [\mathbf{2}]$$

To evaluate the option, we have to solve the *optimal stopping problem*

$$V(x) := \sup_{\tau} E_x[e^{-r\tau}(K - X_\tau)^+]$$

(sup over all stopping times τ and $X_0 = x$ under P_x). **[3]**

The process X satisfying (GBM_r) is specified by a second-order linear differential operator, its (infinitesimal) *generator*,

$$L_X := rxD + \frac{1}{2}\sigma^2 x^2 D^2, \qquad D := \phi/\phi x. \qquad [\mathbf{3}]$$

The closer X gets to 0, the less likely we are to gain by continuing. So we should stop when X gets too small: stop at $\tau = \tau_b := \inf\{t \geq 0 : X_t \leq b\}$ for some $b \in (0, K)$. This gives the following *free boundary problem* for the *unknown value function* $V(x)$ and the *unknown point* b:

$$L_X V = rV \quad \text{for } x > b; \qquad V(x) = (K - x)^+ \qquad \text{for } x = b;$$
$$V'(x) = -1 \qquad \text{for } x = b \text{ (smooth fit)};$$
$$V(x) > (K - x)^+ \qquad \text{for } x > b; \qquad V(x) = (K - x)^+ \qquad \text{for } 0 < x < b. \ [\mathbf{6}]$$

Writing $d := \sigma^2/2$ ('d for diffusion'), $L_X V = rV$ is $dx^2V'' + rxV' - rV = 0$. Trial solution: $V(x) = x^p$. Substituting gives a quadratic for p:

$$p^2 - (1 - \frac{r}{d})p - \frac{r}{d} = 0.$$

The roots are $p = 1$ and $p = -r/d$. So the GS is $V(x) = C_1 x + C_2 x^{-r/d}$. **[5]**
But $V(x) \leq K$ for all $x \geq 0$ (an option giving the right to sell at price K cannot be worth more than K!), so $C_1 = 0$:

$$C_2 = \frac{d}{r}\Big(\frac{K}{1 + d/r}\Big)^{1+r/d}, \qquad b = \frac{K}{1 + d/r}:$$

$$V(x) = \frac{d}{r}\Big(\frac{K}{1 + d/r}\Big)^{1+r/d} x^{-r/d} \quad \text{if } x \in [b, \infty), \qquad K - x \quad \text{if } x \in (0, b]. \ [\mathbf{6}]$$

[Seen – lectures] N. H. Bingham