Math482 EXAMINATION SOLUTIONS 2016

Q1. (i) *Size of trades.*

Small trades do not change the price (appreciably). The price is the level at which markets clear – that is, at which supply and demand balance. If the size of a trade is small, this is negligible compared to the market.

Large trades, by contrast, *do* change prices, as the volume of the trade is no longer negligible. **[6]**

(ii) *Size of market participants.*

A small market participant is a *price taker*. He is able to buy or sell (in the amount relevant to him – small compared to the market) 'without the market noticing', i.e. without changing the price, in any amount he chooses – even a trade large to him will still be small compared to the market. By contrast, large market participants are *price makers*. They execute trades which *are* noticed by the market, and which consequently do shift prices, as they are big enough to alter the balance of supply and demand significantly. **[6]**

(iii) *Size of markets in options and in underlying.*

The underlying is the primary economic asset. Typically, it is a share in a company (part ownership of the company's assets). Or, it may be a commodity (wheat, copper, oil, coffee, etc.). Because a company's shares, or a commodity's price, may go up or down, *options* to buy or sell the underlying at a specified price (the *strike* price K) at or by a specified date (the *expiry* T) are bought and sold. Options serve two basic purposes: speculation (seeking a financial profit, with no interest in the underlying as such) and hedging (seeking insurance against adverse price movement in the underlying). However, options, which have value, are assets in their own right, and can be traded; there are even options on options, etc. The market in options can become much bigger than the market in the underlying. This is an artificial situation, in which the financial aspect becomes divorced from economic reality, leading to *instability* in financial markets. **[7]**

(iv) *To what extent are prices continuous?*

Very large trades cause visible jumps in price, as in (i): they come as *economic shocks* to the market. Small or even medium-sized trades do not cause prices to jump if 'viewed from a distance'. But viewed in enough detail, even trades of normal dize *do* cause prices to jump. Heavily traded stocks under normal market conditions have prices that look continuous from a distance but have many small jumps close up, a phenomenon known as *jitter*. **[6]**

[Largely seen – lectures]

Q2. *Hedging.*
(i) *What is hedging?* Hedging is protecting oneself against loss by buying the opposite of one's position. A *hedging strategy* not only enables one to cover oneself in this way, but also to *price* the option, i.e., the cost of doing so. [**4**]
(ii) *Who hedges, and why?* Hedging is typically engaged in by sellers of options. One sells an option for money, to someone who is buying insurance, and one hopes to make money from it. A seller who remains unhedged has no protection against the loss involved in having the option exercised against him. His position is then *naked*, and this may be too dangerous. [**4**]
(ii) *Types of hedging.* The commonest type of hedging is *delta hedging*, using $\Delta := \partial C / \partial S$. The seller buys enough stock to offset his loss if the option is exercised against him, to first order. Similarly for the other Greeks. [**4**]
(iii) *Discrete v. continuous time; contrasts.* In discrete time, one can hedge in a complete market, but in an incomplete market there may be *unhedgeable risk.* The option seller *re-balances* his portfolio at each time point. [**2**]

In continuous time, this re-balancing is possible in principle. Black-Scholes markets are complete; the driving noise process is Brownian motion (BM); discounted prices are martingales under the EMM, P^* or Q. The Martingale Representation Theorem applies, and shows that option prices can be represented as Brownian integrals. The *integrand* corresponds to the *hedging strategy.* A hedger will need to re-balance continuously. [**2**]

In practice, this cannot be done. For, the sample paths of BM have *infinite variation* (as their quadratic variation is finite, by Lévy's theorem). Not only would re-balancing involve an infinite amount of trading (and so infinite costs, as in reality transaction costs do exist), but would also have to be done extremely roughly. Rebalancing would be like trying to ride a bicycle, following a Brownian-like fractal path – impossible in practice. [**2**]
(iv) *When, or to what extent, should an option seller hedge?*

It depends on how the market moves (are you glad you sold the option or sorry)? To trade, one needs to *take a position* – commit funds, in the presence of uncertainty. One should not do so unless one expects to make money. To trade, one should have a *judgement* of where the market is going, based on knowledge and experience, and be prepared to back it. If the market moves against one, hedge to *unwind one's position* – break even from then on. In any case, one needs to know *how* to do this – just as one needs to know where the (fire or emergency) exit is in a building, plane etc. [**7**]
[Mainly seen in lectures]

Q3. *American options.* The discounting rate per unit time is $1+\rho$. With 'up' and 'down' factors $1+u$, $1+d$ and 'up' and 'down' probabilities q, $1-q$, the discounted price process is a martingale iff $(1+u)q+(1+d)(1-q) = 1+\rho$:

$$uq + d(1-q) = \rho; \qquad (u-d)q = \rho - d : \qquad q = \frac{\rho - d}{u-d}. \qquad \textbf{[3]}$$

To price the American put in this (Cox-Ross-Rubinstein) binomial-tree model:
1. Draw a binary tree showing the initial stock value S and with the right number, N, of time-intervals. **[2]**
2. Fill in the stock prices: after one time interval, these are Su (upper) and Sd (lower); after two, Su^2, Sud and Sd^2; after i time-intervals, $Su^j d^{i-j}$ at the node with j 'up' steps and $i-j$ 'down' steps. **[2]**
3. Using the strike price K and the prices at the *terminal nodes*, fill in the payoffs ($f_{N,j} = \max[K - Su^j d^{N-j}, 0]$) from the option at the terminal nodes (where the values of the European and American options coincide). **[2]**
4. Work back down the tree one time-step. Fill in (a) the 'European' value at the penultimate nodes as the discounted values of the terminal values, under the risk-neutral (P^*, Q) measure – 'q times upper right plus $1-q$ times lower right'; (b) the 'intrinsic' (early-exercise) value; (c) the American put value as the higher of these. **[3]**
5. Treat these values as 'terminal node values', and fill in the values one time-step earlier by repeating Step 4 for this 'reduced tree'. **[2]**
6. Iterate. The value of the American put at time 0 is the value at the root - the last node to be filled in. The 'early-exercise region' is the node set where the early-exercise value is the higher; the rest is the 'continuation region'. **[2]**

Connection with the Snell envelope.

Let $Z = (Z_n)_{n=0}^N$ be the payoff on exercising at time n. To price Z_n, by U_n say, so as to avoid arbitrage: we work backwards in time. Recursively:

$$U_N := Z_N, \qquad U_{n-1} := \max(Z_{n-1}, \frac{1}{1+\rho} E^*[U_n | \mathcal{F}_{n-1}]), \qquad \textbf{[3]}$$

the first alternative on the right corresponding to early exercise, the second to the discounted expectation under P^* (or Q), as usual. Let $\tilde{U}_n = U_n/(1+\rho)^n$ be the discounted price of the American option. Then

$$\tilde{U}_N = \tilde{Z}_N, \qquad \tilde{U}_{n-1} = \max(\tilde{Z}_{n-1}, E^*[\tilde{U}_n | \mathcal{F}_{n-1}]) : \qquad \textbf{[3]}$$

$(\tilde{U}_n)$ is the *Snell envelope* of the discounted payoff process $(\tilde{Z}_n)$. **[3]**
[Seen – lectures]

Q4. (i) *The exponential martingale for Brownian motion.*
The MGF of $X \sim N(\mu, \sigma^2)$ is $E[e^{tX}] = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$, $(*)$, given. For $B = (B_t)$ Brownian motion (BM), write

$$M_t := \exp\{\theta B_t - \frac{1}{2}\theta^2 t\}.$$

Then with $\mathcal{F} = (\mathcal{F}_t)$ the Brownian filtration, for $s \le t$,

$$\begin{aligned} E[M_t|\mathcal{F}_s] &= E[\exp\{\theta B_t - \frac{1}{2}\theta^2 t\}|\mathcal{F}_s] \\ &= E[\exp\{\theta(B_s + (B_t - B_s)) - \frac{1}{2}\theta^2 s - \frac{1}{2}\theta^2(t-s)\}|\mathcal{F}_s] \\ &= \exp\{\theta B_s - \frac{1}{2}\theta^2 s\}.E[\exp\{\theta(B_t - B_s)) - \frac{1}{2}\theta^2(t-s)\}|\mathcal{F}_s], \end{aligned}$$

taking out what is known. The first term on the right is M_s. The conditioning in the second term can be omitted, by independent increments of BM. But $B_t - B_s \sim N(0, t-s)$, which has MGF

$$E[\exp\{\theta(B_t - B_s)\}] = \exp\{\frac{1}{2}\theta^2(t-s)\}$$

(by $(*)$, with $\mu \mapsto 0$, $\theta^2 \mapsto t-s$, $t \mapsto \theta$). So the second term on RHS 1:

$$E[M_t|\mathcal{F}_s] = M_s.$$

So M is a martingale. // **[11]**
(ii) By the normal MGF (given), $M_Y(t) = E[e^{tY}] = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$. Taking $t = 1$, $M_Y(1) = E[e^Y] = \exp\{\mu + \frac{1}{2}\sigma^2\}$. As $X = e^Y$, this gives

$$E[X] = E[e^Y] = e^{\mu + \frac{1}{2}\sigma^2}. \qquad [\mathbf{4}]$$

(iii) In the Black-Scholes model, stock prices are geometric Brownian motions, driven by stochastic differential equations

$$dS = S(\mu dt + \sigma dB), \qquad (GBM)$$

with B Brownian motion. This has solution (quote – Itô's Lemma)

$$S_t = S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\}.$$

So $\log S_t = \log S_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t$ is normal, so S_t is lognormal. **[5]**
(iv) In Girsanov's theorem, we have a process

$$L_t := \exp\{\int_0^t \mu_s dB_s - \frac{1}{2}\int_0^t \mu_s^2 ds\} \qquad (0 \le t \le T),$$

with (μ_s) adapted (with $\int_0^T \mu_s^2 ds < \infty$), and $L = (L_t)$ a martingale. By (i), this martingale condition is satisfied for μ_t constant, identically equal to μ, interpreted as the interest rate – of the risky stock, which Girsanov's theorem transforms by change of measure to r, the riskless interest rate. So (i) enables us to apply Girsanov's theorem, and so obtain the Black-Scholes formula in continuous time. **[5]**
[(i), unseen; (ii), seen – Problems; (iii), (iv), seen, lectures]

Q5. (i) For $s \leq t$,

$$\begin{aligned} E[B_t^2|\mathcal{F}_s] &= E[(B_s + (B_t - B_s))^2|\mathcal{F}_s] \\ &= B_s^2 + B_s E[(B_t - B_s)|\mathcal{F}_s] + E[(B_t - B_s)^2] \\ &\geq B_s^2 + 0 \qquad \text{(as the last term is } \geq 0): \end{aligned}$$

$$E[B_t^2|\mathcal{F}_s] \geq B_s^2,$$

showing that (B_t^2) is a submartingale. **[5]**

(ii) The same calculation shows that for $s \leq t$,

$$\begin{aligned} E[B_t^2|\mathcal{F}_s] &= E[(B_s + (B_t - B_s))^2|\mathcal{F}_s] \\ &= B_s^2 + B_s E[(B_t - B_s)|\mathcal{F}_s] + E[(B_t - B_s)^2] \\ &= B_s^2 + 0 + (t - s) : \end{aligned}$$

$$E[B_t^2 - t|\mathcal{F}_s] = B_s^2 - s,$$

showing that $(B_t^2 - t)$ is a martingale. **[5]**

(iii) Since B_t^2 is a submartingale (i) and t is increasing,

$$B_t^2 = (B_t^2 - t) + t$$

is the Doob-Meyer decomposition of the submartingale B_t^2 (increasing on average), into its martingale part and its increasing part. This identifies t as the quadratic variation of Brownian motion (Lévy's theorem). **[4]**

(iv) One may write Lévy's theorem for $t + dt$ and for t, and subtract, giving (in differential notation)

$$(dB_t)^2 = dt.$$

This is the cornerstone of Itô (or stochastic) calculus, and of Itô's Lemma in particular. **[4]**

(v) Itô calculus and Itô's Lemma, together with *Girsanov's theorem* on change of measure, are the keys to obtaining the *Black-Scholes formula* in continuous time, the central result of option pricing. **[4]**

(vi) Under normal trading conditions, new price-sensitive information arrives (as does the ordinary news) unpredictably (last week's news is no guide to tomorrow's), and the independent-increments assumption is reasonable here. But in a sustained crisis, this breaks down (both last week's and tomorrow's news are on the same crisis), and we need a new model. **[3]**

[Mainly seen – lectures and problems]

Q6. (i) *Sharpe ratio.* The *Sharpe ratio* is $\theta := (\mu - r)/\sigma$: the excess return $\mu - r$ (the investor's reward for taking a risk), compared with the degree of risk as measured by σ. **[3]**

(ii) *Derivation of the Black-Scholes formula via Girsanov's Theorem.*

We summarise the main steps briefly as follows:

(a) Dynamics are given by GBM, $dS_t = \mu S dt + \sigma S dW_t$. **[1]**

(b) Discount: $d\tilde{S}_t = (\mu - r)\tilde{S}dt + \sigma\tilde{S}dW_t = \sigma\tilde{S}(\theta dt + dW_t)$. **[1]**

(c) Use Girsanov's Theorem to change μ to r, so $\theta := (\mu - r)/\sigma$ to 0: under P^*, $d\tilde{S}_t = \sigma\tilde{S}dW_t$. **[2]**

(d) With V the value process, H the strategy, h the payoff, $d\tilde{V}_t(H) = H_t d\tilde{S}_t = H_t.\sigma\tilde{S}dW_t$. Integrate: $\tilde{V}$ gives a P^*-mg, so has constant E^*-expectation. **[2]**

(e) This gives the Risk-Neutral Valuation Formula (RNVF). **[1]**

(f) From RNVF, we can obtain the Black-Scholes formula, by integration. **[1]**

(iii) *Hedging strategy.*

We seek a hedging strategy $H = (H_t^0, H_t)$ (H_t^0 for cash, H_t for stock) that replicates the value process $V = (V_t)$, given by RNVF:

$$V_t = H_t^0 + H_t S_t = E^*[e^{-r(T-t)}h|\mathcal{F}_t]. \qquad [2]$$

Now

$$M_t := E^*[e^{-rT}h|\mathcal{F}_t] \qquad [2]$$

is a (uniformly integrable) martingale under the filtration $\mathcal{F}_t$, that of the driving BM in (GBM), and the filtration is unchanged by the Girsanov change of measure. So by the Representation Theorem for Brownian Martingales, there is some adapted process $K = (K_t)$ with

$$M_t = M_0 + \int_0^t K_s dW_s \qquad (t \in [0, T]). \qquad [2]$$

Take

$$H_t := K_t/(\sigma\tilde{S}_t), \qquad H_t^0 := M_t - H_t\tilde{S}_t : \qquad [2]$$

$$dM_t = K_t dW_t = \frac{K_t}{\sigma\tilde{S}_t}.\sigma\tilde{S}_t dW_t = H_t d\tilde{S}_t, \qquad [2]$$

and the strategy K is self-financing.

(iv) *Limitations.* This is of limited practical value:

(a) the Representation Theorem does not give $K = (K_t)$ explicitly; **[2]**

(b) as Brownian paths have infinite variation, exact hedging in the Black-Scholes model is too rough to be practically possible. **[2]**

[Seen – Lectures] N. H. Bingham