MATH482 SOLUTIONS to MOCK EXAMINATION, 2013

Q1. (i) *Arbitrage.* An arbitrage opportunity is the possibility of making a riskless profit – a trading strategy in which one starts with nothing, never makes a loss, but might make a profit.
(ii) The assumption of absence of arbitrage is unrealistic – but no more so than the other assumptions of a perfect market (same interest rate for borrowing and lending, no liquidity restrictions, no transaction costs, no taxes etc.).

Small arbitrage opportunities may be present and persist. Large ones would attract the attention of speculators and other market participants. This would change the balance of supply and demand, and so prices, so the arbitrage opportunity would shrink (be 'arbitraged away').
(iii) If the market is arbitrage-free,
(a) there exists an equivalent martingale measure;
(b) equilibrium may be possible (it is impossible if arbitrage is present).
(ii) *Completeness.* A market is *complete* if every contingent claim can be replicated, by a combination of stock and cash. Now any such combination may be priced uniquely (count the cash; count the stock; look up the stock price; do the arithmetic). So in a complete market, prices are unique, and do not depend on the attitude to risk of investors (their utility function).

In reality, prices are *not* unique – typically, they fill out an interval, the *bid-ask spread.* This reflects the difference between buying and selling, and the need to have a margin between the two to cover overheads etc.
(iii) *Equivalent martingale measures (EMMs).* We model the uncertainty in risky stocks by a probability measure, P say. As stock prices occur in the real economy, we call P the real(-world) measure. As always, we *discount* prices over time, to work with real prices rather than nominal prices. Two measures P and Q are called *equivalent* if they have the same null sets (sets of probability 0) – i.e., the same things are possible/impossible under both P and Q. We call P^* an/the *equivalent martingale measure* if under P^*, *discounted asset prices become martingales.* The two key results re:
No-arbitrage Theorem. The market has *no arbitrage* (is *NA*) iff EMMs *exist.*
Completeness Theorem. The market is *complete* iff EMMs are *unique.*
(iv) *Risk-neutral valuation.* For complete NA markets, the unique EMM P^* is called the *risk-neutral measure.* The *risk-neutral valuation formula* says that for a complete NA market, asset prices at time t can be calculated as the conditional P^*-expectation of the discounted payoff at expiry T.

Q2. *Perfect Markets.* For simplicity, we shall confine ourselves to option pricing in the simplest (idealised) case, of a *perfect*, or *frictionless*, market. This entails various assumptions:

No transaction costs. We assume that there is no financial friction in the form of transaction costs (one can include transaction costs in the theory, but this is considerably harder).

No taxes. We assume similarly that there are no taxes. We note that a *Tobin tax*, designed partly to damp down excessive volumes of trading and partly to raise money for good causes, has recently been suggested.

Same interest rates for borrowing and for lending. This is clearly unrealistic, as banks make their money on the difference). But it is a reasonable first approximation, and simplifies such things as arbitrage arguments.

Perfect information. We assume that all market participants have perfect information about the past history of price movements, but have no foreknowledge of price-sensitive information (i.e. no insider trading) – also, no information asymmetry, with some participants more knowledgeable than others.

No liquidity restrictions. That is, one can buy or sell unlimited quantities of stock at the currently quoted price. However, in a crisis, no-one wants to trade, and liquidity dries up.

Economic agents are price takers and not price makers. In practice, this is true for small market participants but not for large ones. Big trades do move markets (price is the level at which supply and demand balance; big trades affect this balance significantly).

This restriction emphasizes the difference between Economics and Finance. Much of Economics is concerned with *how prices are arrived at* (supply and demand, etc.). In Finance, at least in this course, we take prices as given.

No restriction on order size; no delay in executing orders. In practice, executing small orders is uneconomic, so there are size limitations. Also, orders are dealt with in job lots, for efficiency. Delays do occur in executing orders, particularly large ones.

No credit risk. Perfect markets assume that all market participants are willing and able to honour their commitments. This ignores the risk of bankruptcy, etc. (necessary, as limited liability is needed to give ordinary market participants the confidence to undertake trade, commerce, investment etc.).

Other risks, e.g.: fraud; human error; insider trading; etc.

Q3. *Put-Call Parity.* The price (or value) of the portfolio at time t is $Ke^{-r(T-t)}$, that is,

$$S + P - C = Ke^{-r(T-t)}.$$

Proof. We prove this by arbitrage. Consider a portfolio which is long one asset, long one put and short one call; write Π (or Π_t) for its value. So

$$\Pi = S + P - C \qquad \text{(S: long asset; P: long put; -C: short call).}$$

Recall that the payoffs at expiry are:

$$\begin{cases} \max(S-K,0) & \text{or} \quad (S-K)_+ \quad \text{for a call,} \\ \max(K-S,0) & \text{or} \quad (K-S)_+ \quad \text{for a put.} \end{cases}$$

So the value of the above portfolio at expiry is K:

$$\begin{cases} S + 0 - (S-K) = K & \text{if } S \geq K \\ S + (K-S) - 0 = K & \text{if } K \geq S, \end{cases}$$

This portfolio thus guarantees a payoff K at time T. How much is it worth at time t?

The riskless way to guarantee a payoff K at time T is to deposit $Ke^{-r(T-t)}$ in the bank at time t and do nothing. If the portfolio is offered for sale at time t too cheaply – at a price $\Pi < Ke^{-r(T-t)}$ – I can *buy* it, *borrow* $Ke^{-r(T-t)}$ from the bank, and pocket a positive profit $Ke^{-r(T-t)} - \Pi > 0$. At time T my portfolio yields K (above), while my bank debt has grown to K. I clear my cash account – use the one to pay off the other – thus locking in my earlier profit, which is *riskless*. If on the other hand the portfolio is offered for sale at time t at too high a price – at price $\Pi > Ke^{-r(T-t)}$ – I can do the exact opposite. I *sell the portfolio short* – that is, I *buy its negative*, long one call, short one put, short one asset, for $-\Pi$, and *invest* $Ke^{-r(T-t)}$ in the bank, pocketing a positive profit $-(-\Pi) - Ke^{-r(T-t)} = \Pi - Ke^{-r(T-t)} > 0$. At time T, my bank deposit has grown to K, and I again clear my cash account – use this to meet my obligation K on the portfolio I sold short, again locking in my earlier riskless profit.

Thus the rational price for the portfolio at time t is *exactly* $Ke^{-r(T-t)}$. *Any other price* presents arbitrageurs with an arbitrage opportunity (to make and lock in a riskless profit) – which they will take! This proves put-call parity.

Note that the value of the portfolio $S + P - C$ is the *discounted value of the riskless equivalent.*

Q4. *Equivalence of American and European calls.*

THEOREM (R. C. Merton, 1973). It is never optimal to exercise an American call option early. That is, the American call option is equivalent to the European call, so has the same value:

$$C = c.$$

First Proof. Consider the following two portfolios:
I: one American call option plus cash Ke^{-rT}; II: one share.
The value of the cash in I is K at time T, $Ke^{-r(T-t)}$ at time t. If the call option is exercised early at $t < T$, the value of Portfolio I is then $S_t - K$ from the call, $Ke^{-r(T-t)}$ from the cash, total

$$S_t - K + Ke^{-r(T-t)}.$$

Since $r > 0$ and $t < T$, this is $< S_t$, the value of Portfolio II at t. So Portfolio I is *always* worth less than Portfolio II if exercised *early.*

If however the option is exercised instead at expiry, T, the American call option is then the same as a European call option. Then (as in Proposition 1 of IV.7): at time T, Portfolio I is worth $\max(S_T, K)$ and Portfolio II is worth S_T. So:

before T, $I < II$,
at T, $I \geq II$ always, and $>$ sometimes.

This direct comparison with the underlying [the share in Portfolio II] shows that early exercise is never optimal. Since an American option at expiry is the same as a European one, this completes the proof. //

Second Proof. Since American options confer all the rights of European options, and more, they must be worth at least as much: $C \geq c$.

Now by Proposition 1 of IV.6, $c_0 \geq S_0 - Ke^{-rT}$. This and $C_0 \geq c_0$ give $C_0 \geq S_0 - Ke^{-rT}$. Using $t < T$ as initial time instead of 0: $C_t \geq S_t - Ke^{-r(T-t)}$. Now $r > 0$ and $t < T$, so $Ke^{-r(T-t)} < K$. This gives

$$C_t > S_t - K.$$

Now if it were optimal to exercise early at $t < T$, the value of the American call (the amount it would yield) would be $S_t - K$. So we would have $C_t = S_t - K$. This would contradict the inequality above, so early exercise can never be optimal. //

Q5. *The Ornstein-Uhlenbeck process.* (i) The Ornstein-Uhlenbeck SDE $dV = -\beta V dt + \sigma dW$ (OU) models the velocity of a diffusing particle. The $-\beta V dt$ term is *frictional drag*; the σdW term is *noise*. **[2]**
(ii) $e^{-\beta t}$ solves the corresponding homogeneous DE $dV = -\beta V dt$. So by variation of parameters, take a trial solution $V = Ce^{-\beta t}$. Then

$$dV = -\beta C e^{-\beta t} dt + e^{-\beta t} dC = -\beta V dt + e^{-\beta t} dC,$$

so V is a solution of (OU) if $e^{-\beta t} dC = \sigma dW$, $dC = \sigma e^{\beta t} dW$, $C = c + \int_0^t e^{\beta u} dW$. So with initial velocity v_0,

$$V = v_0 e^{-\beta t} + \sigma e^{-\beta t} \int_0^t e^{\beta u} dW_u. \qquad [4]$$

(iii) V is Gaussian, as it is obtained from the Gaussian process W by linear operations.
V_t has mean $v_0 e^{-\beta t}$, as $E[e^{\beta u} dW_u] = \int_0^t e^{\beta u} E[dW_u] = 0$.
By the Itô isometry, V_t has variance

$$E[(\sigma e^{-\beta t} \int_0^t e^{\beta u} dW_u)^2] = \sigma^2 \int_0^t (e^{-\beta t + \beta u})^2 du$$

$$= \sigma^2 e^{-2\beta t} \int_0^t e^{2\beta u} du = \sigma^2 e^{-2\beta t} [e^{2\beta t} - 1]/(2\beta) = \sigma^2 [1 - e^{-2\beta t}]/(2\beta).$$

So the limit distribution as $t \to \infty$ is $N(0, \sigma^2/(2\beta))$. **[4]**
(iv) For $u \geq 0$, the covariance is $cov(V_t, V_{t+u})$, which (subtracting off $v_0 e^{-\beta t}$ as we may) is

$$\sigma^2 E[e^{-\beta t} \int_0^t e^{\beta v} dW_v . e^{-\beta(t+u)} (\int_0^t + \int_t^{t+u}) e^{\beta w} dW_w].$$

By independence of Brownian increments, the $\int_t^{t+u}$ term contributes 0, leaving as before

$$cov(V_t, V_{t+u}) = \sigma^2 e^{-\beta u} [1 - e^{-2\beta t}]/(2\beta) \to \sigma^2 e^{-\beta u}/(2\beta) \quad (t \to \infty). \qquad [4]$$

(v) The process V is Markov (a diffusion), being the solution of the SDE (OU). **[3]**
(vi) The process shows *mean reversion*, and the financial relevance is to the *Vasicek model* of interest-rate theory. **[3]**

Q6. *American options – infinite horizon.*

We deal with a *put* option – giving the right to sell at the strike price K, at any time τ of our choosing. This τ has to be a *stopping time*: we decide whether or not to stop at τ based on information already available.

Under the risk-neutral measure, the SDE for GBM becomes

$$dX_t = rX_t dt + \sigma X_t dB_t. \qquad (GBM_r)$$

To evaluate the option, we have to solve the *optimal stopping problem*

$$V(x) := \sup_{\tau} E_x[e^{-r\tau}(K - X_\tau)^+]$$

(sup over all stopping times τ and $X_0 = x$ under P_x).

The process X satisfying (GBM_r) is specified by a second-order linear differential operator, its (infinitesimal) *generator*,

$$L_X := rxD + \frac{1}{2}\sigma^2 x^2 D^2, \qquad D := \partial/\partial x.$$

The closer X gets to 0, the less likely we are to gain by continuing. So we should stop when X gets too small: stop at $\tau = \tau_b := \inf\{t \geq 0 : X_t \leq b\}$ for some $b \in (0, K)$. This gives the following *free boundary problem* for the *unknown value function* $V(x)$ and the *unknown point* b:

$$L_X V = rV \quad \text{for } x > b; \qquad V(x) = (K - x)^+ \qquad \text{for } x = b;$$
$$V'(x) = -1 \qquad \text{for } x = b \text{ (smooth fit)};$$
$$V(x) > (K - x)^+ \qquad \text{for } x > b; \qquad V(x) = (K - x)^+ \qquad \text{for } 0 < x < b.$$

Writing $d := \sigma^2/2$ ('d for diffusion'), $L_X V = rV$ is $dx^2V'' + rxV' - rV = 0$. Trial solution: $V(x) = x^p$. Substituting gives a quadratic for p:

$$p^2 - (1 - \frac{r}{d})p - \frac{r}{d} = 0.$$

One root is $p = 1$; the other is $p = -r/d$. So the general solution is $V(x) = C_1 x + C_2 x^{-r/d}$. But $V(x) \leq K$ for all $x \geq 0$ (an option giving the right to sell at price K cannot be worth more than K!), so $C_1 = 0$:

$$C_2 = \frac{d}{r}\Big(\frac{K}{1 + d/r}\Big)^{1+r/d}, \qquad b = \frac{K}{1 + d/r}.$$

So

$$V(x) = \frac{d}{r}\Big(\frac{K}{1 + d/r}\Big)^{1+r/d} x^{-r/d} \qquad \text{if } x \in [b, \infty), \qquad K - x \qquad \text{if } x \in (0, b].$$