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LONDON TAUGHT COURSE CENTRE: EXAMINATION, 2008
MEASURE-THEORETIC PROBABILITY

Q1. Define the *Poisson point process* $Ppp(\lambda)$ with *intensity* λ.

For $X = (X_t)$ a $Ppp(\lambda)$ and $p \in (0,1)$, the *thinned* process is the process X_p obtained from X by retaining or deleting each of its points with probabilities p and $1-p$, independently of each other. Write $X(A)$ for the number of points of X in a set A, and similarly for X_p.

(i) For an interval I, $X(I) \sim P(\lambda|I|)$ is Poisson distributed with parameter $\lambda|I|$, by definition. Show (by conditioning on $X(I)$, or otherwise) that $X_p(I) \sim P(\lambda p|I|)$.

(ii) Hence or otherwise, show that X_p is a $Ppp(\lambda p)$.

Q2. The *Ornstein-Uhlenbeck process* satisfies the stochastic differential equation (SDE)

$$dV_t = -\beta V_t dt + \sigma dB_t, \qquad (O-U)$$

with $B = (B_t)$ standard Brownian motion.

(i) Interpret $(O-U)$ as modelling the dynamics of the velocity $V = (V_t)$ of a diffusing particle with momentum, subject to (a) a frictional drag, proportional to and opposed to the velocity, and (b) a random noise term, representing the buffeting of the surrounding molecules.

(ii) Show that $V_t = Ce^{-\beta t}$ satisfies the corresponding homogeneous SDE $dV_t = -\beta V_t dt$, for C constant.

(iii) By letting C vary, or otherwise, show that the solution to $(O-U)$ is

$$V_t = \sigma e^{-\beta t} \int_0^t e^{\beta u} dB_u.$$

(iv) Deduce from (iii) that V is Gaussian, and from (i) that V is Markov.

(v) Show that $EV_t = 0$, and that the covariance is given for $s, t \geq 0$ by

$$E[V_t V_{t+s}] = (\sigma^2/2\beta)e^{-\beta s}[1 - e^{-2\beta t}] \to (\sigma^2/2\beta)e^{-\beta s} \qquad (t \to \infty).$$

(vi) Thus as $t \to \infty$ the covariance tends to $(\sigma^2/2\beta)e^{-\beta|\cdot|}$. Deduce that V tends to equilibrium as $t \to \infty$, and that the limit process is stationary Gaussian Markov. Interpret the limiting process physically.

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