LONDON TAUGHT COURSE CENTRE

MEASURE-THEORETIC PROBABILITY

EXAMINATION, 2012

Q1 a. *Probability generating functions (PGFs).* If a random variable X takes values $0, 1, 2, \ldots$ only, write $p_n := P(X = n)$. Then the function $P(s) := E[s^X] = \sum_{n=0}^{\infty} p_n s^n$ is called the *probability generating function* (PGF) of X. Show that
(i) $P'(1) = E[X]$; (ii) $P''(1) = E[X(X-1)]$.

If X, Y are independent with PGFs P, Q, show that $X + Y$ has PGF PQ.

b. *Branching processes.* In a population model, one starts with a single ancestor (the 0th generation). On death, he is replaced by a random number Z of offspring, with PGF $P(s)$ (the 1st generation), and mean $\mu = E[Z]$. They reproduce independently and in the same way, their offspring forming the second generation, with PGF P_2, and so on. Show that:
(i) $P_2(s) = P(P(s))$, the second (functional) iterate of P;
(ii) the nth generation, of size Z_n say, has PGF P_n, the nth functional iterate of P (defined inductively by $P_n = P_{n-1}(P) = P(P_{n-1})$);
(iii) the mean generation size is $E[Z_n] = \mu^n$.

Q2. For $B = (B_t)$ Brownian motion and $M = (M_t)$, where

$$M_t := (B_t^2 - t)^2 - 4\int_0^t B_s^2 ds,$$

(i) find the stochastic differential of M.
(ii) Hence or otherwise, express M as an Itô integral, and show that M is a continuous martingale starting at 0.
(iii) Find the quadratic variation $[M]_t$ of M_t.

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