LTCC: MEASURE-THEORETIC PROBABILITY

EXAMINATION SOLUTIONS, 2011

Q1. (i) For a random variable $X \sim U[0,1]$, take its dyadic expansion $X = \sum_1^\infty \epsilon_n/2^n$. Thus $\epsilon_1 = 0$ iff $X \in [0,1/2)$, 1 iff $X \in [1/2,1)$ (or $[1/2,1]$: we can omit 1, as it carries 0 probability). If $\epsilon_1, \ldots, \epsilon_{n-1}$ are already defined, on the dyadic intervals $[k/2^{n-1},(k+1)/2^{n-1})$, and independent fair coin-tosses (Bernoulli $B(\frac{1}{2})$), split each interval into two halves: $\epsilon_n = 0$ on the left half, 1 on the right half. Then ϵ_n is again $B(\frac{1}{2})$, and is independent of $\epsilon_1, \ldots, \epsilon_{n-1}$:

$$P(\epsilon_n = 0, \epsilon_1 = e_1, \ldots, \epsilon_{n-1} = e_{n-1}) = \frac{1}{2} P(\epsilon_1 = e_1, \ldots, \epsilon_{n-1} = e_{n-1})$$
$$= P(\epsilon_n = 0) P(\epsilon_1 = e_1, \ldots, \epsilon_{n-1} = e_{n-1}),$$

and similarly for $\epsilon_n = 1$. By induction, ϵ_n $(n = 1, 2, \ldots)$ are independent $B(\frac{1}{2})$. Conversely, given ϵ_n independent coin tosses, form $X := \sum_1^\infty \epsilon_n/2^n$. Then $X_n := \sum_1^n \epsilon_k/2^k \to X$ a.s. The distribution function F_n of X_n has jumps $1/2^n$ at $k/2^n$, $k = 0, 1, \ldots, 2^n - 1$. This 'saw-tooth jump function' converges to x on $[0,1]$, the distribution function of $U[0,1]$ ($\sup |F_n(x) - x| = 2^{-n} \to 0$ as $n \to \infty$). So $X \sim U[0,1]$. So if $X = \sum_1^\infty \epsilon_n/2^n$, $X \sim U[0,1]$ iff ϵ_n are independent coin tosses – the Lebesgue probability space models *both* (a) length on the unit interval *and* (b) infinitely many independent coin tosses.

(ii) From the given $U[0,1]$, we get by dyadic expansion as above a sequence of independent coin-tosses ϵ_n. Rearrange these into a two-suffix array ϵ_{jk} (as with Cantor's proof of 1873 that the rationals are countable). The ϵ_{jk} are all independent, so the $X_j := \sum \epsilon_{jk}/2^k$ are independent, and $U[0,1]$ by above. So from *one* $U(0,1)$, we get in this way *infinitely many copies*.

(iii) The standard normal distribution function Φ is continuous and strictly increasing, so its inverse function $\Phi^{-1}(t) := \inf\{\Phi(x) \geq t\}$ $(0 < t < 1)$ is continuous and strictly increasing. Then if $U \sim U[0,1]$, $X := \Phi^{-1}(U) \sim \Phi$: for, $\{X \leq x\} = \{\Phi^{-1}(U) \leq x\} = \{U \leq \Phi(x)\}$, which has probability $\Phi(x)$ as U is uniform. Hence by (ii) above we can then generate infinitely many independent standard normals. We can hence simulate a Brownian motion $B = (B_t)$ from $B_t = \sum_0^\infty \lambda_n Z_n \Delta_n(t)$, with Z_n independent standard normals, $\Delta_n(t)$ the Schauder functions and λ_n suitable normalising constants.

(iv) Similarly, given one $U[0,1]$, we can split it as in (ii) into infinitely many independent $U[0,1]$s. From each, we can generate a Brownian motion by (iii), giving infinitely many independent Brownian motions.

Q2. (i) The *Itô isometry* states that for $f \in H^2 := H^2(0,T) := \{f : E[\int_0^T f^2(\omega,t)dt] < \infty\}$,

$$E[(\int_0^t f^2(\omega,u)dB_u)^2] = E[\int_0^t f^2(\omega,u)du].$$

(ii) *Conditional Itô isometry.* For $0 \leq s \leq t \leq T$,

$$E[(\int_s^t f^2(\omega,u)dB_u)^2|\mathcal{F}_s] = E[\int_s^t f^2(\omega,u)du|\mathcal{F}_s].$$

Proof. It suffices to show that for all $A \in \mathcal{F}_s$,

$$E[I(A)(\int_s^t f^2(\omega,u)dB_u)^2] = E[I(A)\int_s^t f^2(\omega,u)du].$$

This follows from the unconditional Itô isometry, applied to the integrand $g(\omega,u) := fI_A(\omega)I_{(s,t]}(u)$.
(iii) For $s \leq t$, $M_t := (\int_s^t f(\omega,u)dB_u)^2 - \int_0^t f^2(\omega,u)du$,

$$E[M_t|\mathcal{F}_s] = E[\{(\int_0^s + \int_s^t)f_u dB_u\}^2|\mathcal{F}_s] - \int_0^s f_u^2 du - E[\int_s^t f_u^2 du|\mathcal{F}_s]$$

$$= E[(\int_0^s f_u dB_u)^2] + 2(\int_0^s f_u dB_u)E[\int_s^t f_u dB_u|\mathcal{F}_s] + E[(\int_s^t f_u dB_u)^2|\mathcal{F}_s] - \int_0^s f_u^2 du - E[\int_s^t f_u^2 du|\mathcal{F}_s].$$

The first and fourth terms give M_s. The third and fifth terms cancel, by the conditional Itô isometry (ii). The second factor in the second term involves an Itô integral, which (for an integral $f \in H^2$) is a martingale, so has constant expectation, which is 0 on taking $t = s$, so the second term is 0. Combining, the RHS is M_s, which proves that M is a martingale.
(iv) Taking $f \equiv 1$ gives $M_t := B_t^2 - t$ is a martingale.

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