

**LONDON TAUGHT COURSE CENTRE**  
**MEASURE-THEORETIC PROBABILITY**  
**EXAMINATION SOLUTIONS, 2014**

Q1 *Brownian covariance.* For  $s \leq t$ ,

$$B_t = B_s + (B_t - B_s), \quad B_s B_t = B_s^2 + B_s(B_t - B_s).$$

Take expectations: on the left we get  $\text{cov}(B_s, B_t)$ . The first term on the right is, as  $E[B_s] = 0$ ,  $\text{var}(B_s) = s$ . As BM has independent increments,  $B_t - B_s$  is independent of  $B_s$ , so

$$E[B_s(B_t - B_s)] = E[B_s] \cdot E[B_t - B_s] = 0 \cdot 0 = 0.$$

Combining,  $\text{cov}(B_s, B_t) = s$  for  $s \leq t$ . Similarly, for  $t \leq s$  we get  $t$ . Combining,  $\text{cov}(B_s, B_t) = \min(s, t)$ .

Q2 *Brownian scaling.* With  $B_c(t) := B(c^2 t)/c$ ,

$$\text{cov}(B_c(s), B_c(t)) = E[B(c^2 s)/c \cdot B(c^2 t)/c] = c^{-2} \min(c^2 s, c^2 t) = \min(s, t) = \text{cov}(B_s, B_t).$$

So  $B_c$  has the same mean 0 and covariance  $\min(s, t)$  as BM. It is also (from its definition) continuous, Gaussian, stationary independent increments etc. So it has all the defining properties of BM. So it *is* BM.

So BM is a *fractal*: it reproduces itself if time and space are scaled together in this way. This is why if we "zoom in and blow up" a Brownian path, it still looks like a Brownian path – however often we do this. By contrast, if we zoom in and blow up a smooth function, it starts to look straight (because it has a tangent).

Specialising to the zero set  $Z$  of BM  $B$ , this too is a fractal because  $B$  is.

Q3. We calculate  $\int B(u)dB(u)$ . We start by approximating the integrand by a sequence of simple functions.

$$X_n(u) = \begin{cases} B(0) = 0 & \text{if } 0 \leq u \leq t/n, \\ B(t/n) & \text{if } t/n < u \leq 2t/n, \\ \vdots & \vdots \\ B((n-1)t/n) & \text{if } (n-1)t/n < u \leq t. \end{cases}$$

By definition,

$$\int_0^t B(u)dB(u) = \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} B(kt/n)(B((k+1)t/n) - B(kt/n)).$$

Replacing  $B(kt/n)$  by  $\frac{1}{2}(B((k+1)t/n) + B(kt/n)) - \frac{1}{2}(B((k+1)t/n) - B(kt/n))$ , the RHS is

$$\begin{aligned} & \sum \frac{1}{2}(B((k+1)t/n) + B(kt/n)) \cdot (B((k+1)t/n) - B(kt/n)) \\ & - \sum \frac{1}{2}(B((k+1)t/n) - B(kt/n)) \cdot (B((k+1)t/n) - B(kt/n)). \end{aligned}$$

The first sum is  $\sum \frac{1}{2}(B((k+1)t/n)^2 - B(kt/n)^2)$ , which telescopes (as a sum of differences) to  $\frac{1}{2}B(t)^2$  ( $B(0) = 0$ ). The second sum is  $\frac{1}{2}\sum (B((k+1)t/n) - B(kt/n))^2$ , an approximation to the quadratic variation of  $B$  on  $[0, t]$ , which tends to  $\frac{1}{2}t$  by Lévy's theorem on the QV. Combining,

$$\int_0^t B(u)dB(u) = \frac{1}{2}B(t)^2 - \frac{1}{2}t.$$

Note the contrast with ordinary (Newton-Leibniz) calculus! Itô calculus requires the second term on the right – the Itô correction term – which arises from the quadratic variation of  $B$ .

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