LONDON TAUGHT COURSE CENTRE

MEASURE-THEORETIC PROBABILITY

SOLUTIONS TO EXAMINATION, 2013

Q1. Write $\sigma_n(X) := \sigma(X_1, \ldots, X_n)$. Then $\sigma(X)$ is the σ-field generated by the increasing family $\sigma_n(X)$ of σ-fields. The union $\cup_n \sigma_n(X)$ forms a field, which generates the σ-field $\sigma(X)$. So (from the Carathéodory extension procedure, given), for $A \in \sigma(X)$ there are $A_n \in \sigma_n(X)$ with

$$P(A \Delta A_n) \to 0, \quad i.e. \quad P(A \setminus A_n) \to 0, \quad P(A_n \setminus A) \to 0$$

(Δ is the symmetric difference). So (writing '$o(1)$' for 'term tending to 0')

$$P(A_n) = P(A_n \cap A) + P(A_n \setminus A) = P(A_n \cap A) + o(1),$$

and similarly

$$P(A) = P(A \cap A_n) + P(A \setminus A_n) = P(A_n \cap A) + o(1) = P(A_n) + o(1).$$

If $A \in \mathcal{T}$ is a tail event, A depends only on random variables X_k sufficiently far along (i.e. for k sufficiently large). As the X_n are independent, A is independent of each $\sigma_n(X)$. So

$$P(A \cap A_n) = P(A).P(A_n).$$

Let $n \to \infty$: by above, we get

$$P(A) = P(A).P(A) = P(A)^2.$$

So $x = P(A)$ satisfies the equation $x = x^2$, i.e. $x^2 - x = x(x-1) = 0$, whose roots are $x = 0$ or 1. So $P(A) = 0$ or 1: the probability of a tail event of a sequence of independent random variables is 0 or 1, proving Kolmogorov's Zero-One Law. //

If the events A_n are independent, their indicators I_{A_n} are independent random variables. The event

$$A := \text{limsup} A_n := \cap_n \cup_{m=n}^{\infty} A_m = \{A_n \ i.o.\}$$

that infinitely many A_n occur is a tail event. By Kolmogorov's Zero-One Law above, $P(A) = 0$ or 1. By the Borel-Cantelli Lemmas, $P(A) = 0$ or 1 according as $\sum P(A_n)$ converges or diverges.

Q2. (i) For $s < t$, $M_s = E[M_t|\mathcal{F}_s]$ as M is a martingale. So by the conditional Jensen inequality,

$$\phi(M_s) = \phi(E[M_t|\mathcal{F}_s]) \leq E[\phi(M_t)|\mathcal{F}_s],$$

which says that $\phi(M)$ is a submartingale.
(ii) If M is a submartingale, $M_s \leq E[M_t|\mathcal{F}_s]$. As ϕ is non-decreasing on the range of M,

$$\begin{aligned} \phi(M_s) &\leq \phi(E[M_t|\mathcal{F}_s]), \\ &\leq E[\phi(M_t)|\mathcal{F}_s] \end{aligned}$$

by the conditional Jensen inequality again, and again $\phi(M)$ is a submartingale.
(iii) As Brownian motion B is a martingale (lectures), and x^2 is convex (its second derivative is $1 \geq 0$), B^2 is a submartingale by (i).
(iv) For $s < t$,

$$\begin{aligned} E[B_t^2 - t|\mathcal{F}_s] &= E[\{(B_t - B_s) + B_s\}^2 - t|\mathcal{F}_s] \\ &= E[(B_t - B_s)^2|\mathcal{F}_s] + 2E[(B_t - B_s)B_s|\mathcal{F}_s] + E[B_s^2|\mathcal{F}_s]. \end{aligned}$$

The first term is $E[B_{t-s}^2] = t - s$, by the strong Markov property of BM. The second term is $2B_sE[B_t - B_s|\mathcal{F}_s] = 0$; the third term is B_s^2. Combining,

$$E[B_t^2 - t|\mathcal{F}_s] = B_s^2 - s,$$

so $B_t^2 - t$ is a martingale.
(v) By (iv),

$$B_t^2 = [B_t^2 - t] + t \qquad (\text{submg} = \text{mg} + \text{incr})$$

is the Doob-Meyer decomposition of B_t^2. The increasing process here is t, which is thus the quadratic variation of Brownian motion B.

N. H. Bingham