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LONDON TAUGHT COURSE CENTRE:
SOLUTIONS TO EXAMINATION, 2008
MEASURE-THEORETIC PROBABILITY

Q1. The process X is a *Poisson point process* with *intensity* λ if for any measurable set (equivalently, for any interval) A, the number $X(A)$ of points of the point process X in the set A is Poisson with parameter $\lambda|A|$ ($|.|$ is Lebesgue measure), and for disjoint sets A, B, the counts $X(A)$ and $X(B)$ are independent.
(i)

$$P(X_p(I) = k) = \sum_{n=k}^{\infty} P(X_p(I) = k|X(I) = n)P(X(I) = n).$$

Now $P(X(I) = n) = e^{-\lambda|I|}(\lambda|I|)^n/n!$, as $X(I) \sim P(\lambda|I|)$, and the conditional probability is binomial:

$$P(X_p(I) = k|X(I) = n) = \binom{n}{k} p^k(1-p)^{n-k}.$$

So the first RHS is

$$\sum_{n=k}^{\infty} \binom{n}{k} p^k(1-p)^{n-k}.e^{-\lambda|I|}(\lambda|I|)^n/n! = \sum_{n=k}^{\infty} e^{-\lambda|I|}(\lambda|I|)^n.p^k(1-p)^{n-k}/k!(n-k)!,$$

or writing $n - k = j$,

$$\frac{\lambda^k|I|^k p^k e^{-\lambda|I|}}{k!} \sum_{j=0}^{\infty} \lambda^j|I|^j(1-p)^j/j! = e^{-\lambda|I|}\frac{\lambda^k p^k|I|^k}{k!}.e^{\lambda|I|(1-p)} = e^{-\lambda|I|p}(\lambda|I|p)^k/k!$$

So $X_p(I)$ is $P(\lambda p|I|)$, as required.
(ii) Because X is a *Ppp*, the numbers of points of X in disjoint intervals are independent. They are still independent after independent thinning with probability p. From this and (i), X_p is a $Ppp(\lambda p)$.

Q2. (i) Newton's Law of Motion is force = mass times acceleration, and acceleration is the rate of change of velocity V. A diffusing particle with momentum (a Brownian particle does not have momentum!) will be acted on by two forces, frictional drag and random bombardment, and the SDE $(O-U)$ expresses Newton's Law of Motion for this situation.
(ii) Immediate, by calculus.
(iii) If $V = Ce^{-\beta t}$, $dV = e^{-\beta t}dC - C.\beta e^{-\beta t}dt$. So V satisfies $(O-U)$, i.e. $dV = -\beta V_t dt + \sigma dB_t = -C.\beta e^{-\beta t}dt + \sigma dB_t$, if $e^{-\beta t}dC = \sigma dB$, $dC = e^{\beta t}\sigma dB$, $C_t = \sigma \int_0^t e^{\beta u}dB_u$,

$V_t = \sigma e^{-\beta t} \int_0^t e^{\beta u} dB_u$.
(iv) Linear combinations of independent Gaussians are Gaussian. Limits of Gaussians are Gaussian. Combining, integrals with Brownian integrands – i.e., Itô integrals – are Gaussian. The solution to the SDE is a diffusion, so (path-continuous and strong) Markov.
(v) $EV_t = e^{-\beta t} \int_0^t e^{\beta u} dEB_u = 0$, as $EB_u = 0$. For $s, t \geq 0$,

$$V_t V_{t+s} = \sigma^2 e^{-\beta(2t+s)}[(\int_0^t e^{\beta u} dB_u)^2 + \int_0^t e^{\beta u} dB_u . \int_t^{t+s} e^{\beta v} dB_v].$$

Take expectations left and right. Take E inside the integrals in the second term on the right. We get $EdB_u dB_v = EdB_u.EdB_v = 0.0 = 0$, since the ranges of integration are disjoint, so the Brownian increments dB_u, dB_v are independent. Similarly, in the first term on the right (replacing the u in the second factor by v), $EdB_u dB_v = EdB_u.EdB_v = 0.0 = 0$, except when $u = v$, when $(dB_u)^2 = du$, so $E[(dB_u)^2] = du$. This reduces the double (or repeated) integral to a single integral, giving

$$E[V_t V_{t+s}] = \sigma^2 e^{-\beta(2t+s)} \int_0^t e^{2\beta u} du = \sigma^2 e^{-\beta(2t+s)}[e^{2\beta t} - 1]/(2\beta) = \sigma^2 e^{-\beta s}[1 - e^{-2\beta t}]/2\beta$$

$$\to (\sigma^2/2\beta) e^{-\beta s} \qquad (t \to \infty).$$

(vi) The limiting covariance $(\sigma^2/2\beta)e^{-\beta s}$ follows from this, and the limiting mean is 0. So the finite-dimensional distributions are Gaussian with this mean and covariance. As a Gaussian process is determined by its mean and covariance, there is a process with these finite-dimensional distributions, and this process is stationary as the covariance depends only on time *difference*, not time. The physical interpretation is that as time passes, the diffusing particle settles down to a steady state, or reaches equilibrium (its velocity distribution is Gaussian – it is the *Maxwell-Boltzmann distribution* of Statistical Mechanics).

NHB