LONDON TAUGHT COURSE CENTRE

MEASURE-THEORETIC PROBABILITY

EXAMINATION, 2011

Q1. (i) For $X \in [0,1]$, let its *dyadic expansion* be $X = \sum_{n=1}^{\infty} \epsilon_n/2^n$, $\epsilon_n \in \{0,1\}$. Show that X has the uniform distribution on $[0,1]$, $X \sim U[0,1]$, if and only if the ϵ_n are independent tosses of a fair coin (independent $B(\frac{1}{2})$ – Bernoulli distributed with parameter 1/2).
(ii) By (i) and a diagonalisation argument, or otherwise, given $U \sim U[0,1]$ show how to generate an infinite sequence of independent copies $U_n \sim U[0,1]$.
(iii) For Φ the standard normal distribution function, with inverse function Φ^{-1}, show that for $U \sim U[0,1]$, $\Phi^{-1}(U) \sim \Phi$. Hence, given one $U \sim U[0,1]$, show how to construct a Brownian motion.
(iv) Hence or otherwise, given one $U \sim U[0,1]$, show how to construct infinitely many independent Brownian motions.

Q2. (i) Define the space $H^2 := H^2(0,T)$, and state without proof the Itô isometry for Itô integrals with integrands in H^2.
(ii) Prove the *conditional Itô isometry*: for $0 \leq s \leq t \leq T$, $f \in H^2$,

$$E[(\int_s^t f^2(\omega,u)dB_u)^2|\mathcal{F}_s] = E[\int_s^t f^2(\omega,u)du|\mathcal{F}_s].$$

(iii) Show that for $f \in H^2$

$$M_t := (\int_0^t f(\omega,u)dB_u)^2 - \int_0^t f^2(\omega,u)du$$

is a martingale.
(iv) Deduce that $B^2(t,\omega) - t$ is a martingale.

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