LONDON TAUGHT COURSE CENTRE

MEASURE-THEORETIC PROBABILITY

EXAMINATION, 2014

Q1 *Brownian covariance.* The *covariance* of random variables X, Y is

$$cov(X,Y) := E[(X - E[X])(Y - E[Y])].$$

Show that for $B = (B_t)$ Brownian motion (BM), its covariance is

$$cov(B_s, B_t) = min(s,t).$$

We quote that for a Gaussian process (one all of whose finite-dimensional distributions are Gaussian, such as BM), the process is characterised by its mean function and covariance function (so mean 0 and covariance $\min(s,t)$ characterise BM).

Q2 *Brownian scaling.* With $c > 0$ and B Brownian motion, show that B_c, where

$$B_c(t) := B(c^2 t)/c,$$

has the same covariance function $\min(s,t)$ as Brownian motion B. Deduce that (as B_c is also continuous and Gaussian) that B_c *is* Brownian motion. It is formed from B by *Brownian scaling.*

Deduce that B is *self-similar*: it reproduces itself it time and space are both scaled as above. We call such a self-similar process a *fractal.*

If Z is the zero-set of B and Z_c that of B_c, deduce that Z also is a fractal.

Q3. By writing

$$\int_0^t B(u)dB(u) = \lim_{n\to\infty} \sum_{k=0}^{n-1} B(kt/n)(B((k+1)t/n) - B(kt/n))$$

$$= \lim \sum \frac{1}{2}(B((k+1)t/n) + B(kt/n)).(B((k+1)t/n) - B(kt/n))$$

$$- \lim \sum \frac{1}{2}(B((k+1)t/n) - B(kt/n)).(B((k+1)t/n) - B(kt/n)),$$

or otherwise, show that

$$\int_0^t B(u)dB(u) = \frac{1}{2}B(t)^2 - \frac{1}{2}t.$$

Comment on the difference between this Itô calculus result and ordinary (Newton-Leibniz) calculus.

NHB