**LONDON TAUGHT COURSE CENTRE**

**MEASURE-THEORETIC PROBABILITY**

**EXAMINATION, 2013**

Q1. The *tail $\sigma$-field* $\mathcal{T}$ of a process $X = (X_n)$ is the sub-$\sigma$-field of $\sigma(X)$ invariant under changes to finitely many of the $X_n$.

Prove *Kolmogorov's Zero-One Law*: that if the $X_n$) are independent, the probability of a tail event is 0 or 1. (You may quote that, from the Carathéodory extension procedure, if a measure $\mu$ is extended from a field $\mathcal{F}_0$ to the generated $\sigma$-field $\mathcal{F}$, $\mu$ may be approximated on $\mathcal{F}$ by its values on $\mathcal{F}_0$ in the sense that for any $A \in \mathcal{F}$ and $\epsilon > 0$ there is a set $A_0 \in \mathcal{F}_0$ such that $\mu(A \Delta A_0) < \epsilon$.)

If $A_n$ are independent events, and $A := \text{limsup} A_n$ is the event that infinitely many of the $A_n$ occur, deduce that $A$ has probability 0 or 1.

State, without proof, when $P(A)$ is 0 and when it is 1.

Q2. A function $\phi$ is *convex* on a set $A$ if

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y) \qquad \forall \lambda \in [0,1], x, y \in A.$$

*Jensen's inequality* states that

$$\phi(E[X]) \leq E[\phi(X)]$$

for convex $\phi$ and random variables $X$ with $X$, $\phi(X)$ both integrable. The *conditional Jensen inequality* states that for $\mathcal{A}$ a $\sigma$-field, $\phi$, $X$ as above,

$$\phi(E[X|\mathcal{A}]) \leq E[\phi(X)|\mathcal{A}].$$

(i) If $M = (M_t)$ is a martingale, and $\phi$ is a convex function such that each $\phi(M_t)$ is integrable, show that $\phi(M)$ is a submartingale.
(ii) If in (i) $M$ is a submartingale, and $\phi$ is also non-decreasing on the range of $M$, show that again $\phi(M)$ is a submartingale.
(iii) Deduce that $B^2 = (B_t^2)$ is a submartingale.
(iv) Show that for $B = (B_t)$ Brownian motion, $B_t^2 - t$ is a martingale.
(v) Find the increasing process in its Doob-Meyer decomposition. Deduce that Brownian motion has quadratic variation $t$.

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