LONDON TAUGHT COURSE CENTRE

MEASURE-THEORETIC PROBABILITY

EXAMINATION SOLUTIONS, 2012

Q1. (a) PGFs. $P(s) = E[s^X] = \sum_0^\infty P(X = n)s^n = \sum_0^\infty p_n s^n$.
As $P(1) = \sum_0^\infty P(X = n) = 1$ converges, the radius of convergence R of the power series is at least 1.

$$P'(s) = \sum_0^\infty P(X = n).ns^{n-1} = \sum_0^\infty np_n s^{n-1},$$

so if $R > 1$ we can substitute to get $P'(1) = \sum nP(X = n) = E[X]$, and the same is true for $R = 1$ by Abel's Continuity Theorem. **[4]**
Similarly, $P''(1) = \sum n(n-1)P(X = n)$, so

$$P''(1) = \sum n(n-1)P(X = n) = E[X(X-1)]. \qquad \textbf{[3]}$$

If X, Y are independent, $X + Y$ has PGF $R(s) := E[s^{X+Y}] = E[s^X.s^Y]$ (property of exponentials). As s^X, s^Y are independent, the Multiplication Theorem gives $R(s) = E[s^X].E[s^Y] = P(s).Q(s)$. **[3]**
(b) Branching processes.
(i) Z_2 is the sum of a random number, Z_1, of independent copies of Z. So

$$P_2(s) := E[s^{Z_2}] = \sum_{k=0}^\infty E[s^{Z_2}|Z_1 = k]P(Z_1 = k).$$

Now when $Z_1 = k$, Z_2 is a sum of k independent copies of Z, each with PGF $P(s)$, so by (a) has (conditional) PGF $P(s)^k$. So

$$P_2(s) = \sum_0^\infty p_k P(s)^k = P(P(s)). \qquad \textbf{[4]}$$

(ii) Similarly, or by induction on n, Z_n has PGF P_n. **[3]**
(iii)

$$P_n'(s) = P'(P_{n-1}(s)).P_{n-1}'(s).$$

So letting $s = 1$ ($R > 1$), or $s \uparrow 1$ ($R = 1$) and using Abel's Continuity Theorem, since P_{n-1}, being a PGF, has value 1 at 1, $P_n'(1) = P'(1).P_{n-1}'(1) = \mu.P_{n-1}'(1)$, so by induction

$$P_n'(1) = \mu^n : \quad E[Z_n] = \mu^n. \qquad \textbf{[3]}$$

Q2. (i). Write $f(B,t) := (B^2 - t)^2$. By Itô's formula,

$$df = f_B dB + f_t dt + \frac{1}{2}[f_{BB}(dB)^2 + 2f_{Bt} dB dt + f_{tt}(dt)^2].$$

In the [...] on RHS, $(dB)^2 = dt$, $dBdt = 0$, $(dt)^2 = 0$. Also

$$f_B = 2.2B(B^2 - t), \quad f_t = -2(B^2 - t), \quad f_{BB} = 4(B^2 - t) + 4B.2B = 12B^2 - 4t.$$

So

$$df = 4B(B^2 - t)dB - 2(B^2 - t)dt + (6B^2 - 2t)dt = 4B(B^2 - t)dB + 4B^2 dt.$$

As $M = f - 4\int_0^t B_s^2 ds$, the stochastic differential of M is

$$dM = df - 4B_t^2 dt = 4B(B^2 - t)dB. \qquad \textbf{[6]}$$

(ii) So integrating, M is the Itô integral

$$M_t = 4\int_0^t B_s(B_s^2 - s) dB_s. \qquad \textbf{[4]}$$

The Itô integral on the RHS is a continuous local martingale starting from 0. Now $B_t =_d t^{1/2}.Z$ where Z is $N(0,1)$. As Z has all moments finite, each $E[B_t^n]$ is a polynomial in t. So the integrand $h = h(B_t, t)$ on RHS satisfies the integrability condition $\int_0^t E[h_s^2] ds < \infty$ for all t. So the RHS is a (true) continuous mg starting from 0. **[4]**

(iii) With $([M_t])$ the quadratic variation of M,

$$d[M]_t = (dM)_t^2; \qquad dM_t = 4B_t(B_t^2 - t)dB_t.$$

So

$$d[M]_t = 16B_t^2(B_t^2 - t)^2(dB_t)^2 = 16B_t^2(B_t^2 - t)^2 dt :$$

$$[M]_t = 16\int_0^t B_s^2(B_s^2 - s)^2 ds. \qquad \textbf{[6]}$$

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