LTCC, MEASURE-THEORETIC PROBABILITY: EXAMINATION SOLUTIONS 2018

Q1. (a) If the random variable U has the uniform distribution $U[0,1]$ ($U \sim U[0,1]$): let Φ be the distribution function of the standard normal distribution $N(0,1)$. Then Φ is continuous and strictly increasing, and so has an inverse function Φ^{-1}, also continuous and strictly increasing. Then writing $Z := \Phi^{-1}U$,

$$P(Z \leq x) = P(\Phi^{-1}(U) \leq x) = P(U \leq \Phi(x)) = \Phi(x)$$

(as $\Phi(x) \in (0,1)$: this is the 'probability integral transformation'). So, writing

$$Z_n := \Phi^{-1}(U_n),$$

the Z_n are independent (as the U_n are) and standard normal (as above).
(b) For $n \geq 1$ integer, write $n = 2^j + k$, with $j, k \geq 0$ integers and $k < 2^j$. Define

$$\Delta(t) := 2t \quad (0 \leq t \leq \frac{1}{2}), \quad 2(1-t) \quad (\frac{1}{2} \leq t \leq 1), \quad 0 \text{ else}.$$

Write $\Delta_0(t) := t$, $\Delta_1(t) := \Delta(t)$, and use Δ as mother wavelet to define daughter wavelets

$$\Delta_n(t) := \Delta(2^j t - k) \qquad (n = 2^j + k),$$

the Schauder system (so Δ_n has support the dyadic interval $[k/2^j, (k+1)/2^j]$). Then with the appropriate normalisation constants

$$\lambda_0 = 1, \qquad \lambda_n = \frac{1}{2}.2^{-j/2}$$

the Δ_n form a complete orthonormal system (cons) on $L_2[0,1]$. Then by the Paley-Wiener-Zygmund (PWZ) theorem, if

$$B(t) := \sum_{n=0}^{\infty} \lambda_n Z_n \Delta_n(t),$$

$B = (B(t))$ is a Brownian motion, the series being uniformly convergent on $[0,1]$, a.s.

Q2. The uniform distribution $U[0,1]$ corresponds to an infinite sequence of independent coin-tosses: if the ϵ_n are independent, taking values 0, 1 with probability $\frac{1}{2}$, and

$$U := \sum_{n=1}^{\infty} \epsilon_n,$$

then $U \sim U[0,1]$, and conversely, if $U \sim U[0,1]$, its dyadic expansion coefficients ϵ_n as above are independent coin-tosses.

As in the Cantor diagonalisation procedure, showing that the rationals are countable, we can split the single sequence (ϵ_n) of independent coin-tosses into *infinitely many* sequences (still necessarily of independent coin-tosses). As in Q1, this gives us a Brownian motion.

We can repeat this splitting process, and split each sequence so obtained into infinitely many sequences also. Each can be used to generate a Brownian motion, as above, giving the required infinitely many independent Brownian motions.

So although on the face of it the PWZ theorem needs as its 'raw material' an infinite sequence of standard normals (equivalently, uniforms), one can 'economise', and use only *one* (uniform, say).

Q3. The Brownian bridge B_0 on $[0,1]$ is obtained from Brownian motion B by

$$B_0(t) := B(t) - tB(1).$$

So, as $\lambda_0 = 1$ and both Z_0 and $B(1)$ are standard normal, the $tB(1)$ term here may be taken as the $n = 0$ term in the PWZ expansion. Thus the Brownian-bridge case can be obtained from the Brownian case above by dropping all the $n = 0$ terms.

Q4. As with everything else in Analysis, which deals with real numbers (whose decimal expansions terminate or recur iff the number is *rational*, and 'rationals are flukes' – only countably many), one has to *truncate*, to conform to the limitations of computer power or the needs of the investigation. This is not problematic here, as the PWZ expansion is *uniformly* convergent, a.s. – so the truncation needed at one point will suffice for all.

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