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LONDON TAUGHT COURSE CENTRE: SOLUTIONS TO MOCK EXAMINATION, 2008 MEASURE-THEORETIC PROBABILITY

Q1. A.

(i) $P_{X+Y}(s) = \sum_{n=0}^{\infty} P(X+Y=n)s^n$. $P(X+Y=n) = \sum_{k=0}^{n} P(X=k, Y=n-k) = \sum_{k=0}^{n} P(X=k)P(Y=n-k)$, by independence. Substitute, and put $j := n-k$ to get

$$P_{X+Y}(s) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} P(X=k).P(Y=j).s^j.s^k$$

$$= \sum_{k=0}^{\infty} P(X=k)s^k . \sum_{j=0}^{\infty} P(Y=j).s^j = P_X(s).P_Y(s).$$

(ii) $P_X(s) = \sum_{n=0}^{\infty} e^{-\lambda}(\lambda^n/n!).s^n = e^{-\lambda}\sum_{n=0}^{\infty}(\lambda s)^n/n! = e^{-\lambda}.e^{\lambda s} = e^{-\lambda(1-s)}$.

(iii) Combining, $P_{X+Y}(s) = e^{-\lambda(1-s)}.e^{-\mu(1-s)} = e^{-(\lambda+\mu)(1-s)}$. So $X+Y \sim P(\lambda+\mu)$.

(iv) $EX = \sum_{n=0}^{\infty} n.P(X=n)$. One can evaluate the sum directly, but the easiest way to get the sum is to differentiate the generating function and evaluate it at $s=1$ (proof: one can take the d/ds inside the sum; $d(s^n)/ds = ns^{n-1}$; this gives the factor n on putting $s=1$). As $d[e^{-\lambda(1-s)}]/ds = \lambda e^{-\lambda(1-s)}$, this gives $EX = \lambda$: the mean of a Poisson random variable is its parameter (its variance is λ too, but we won't need this here). Then $E(X+Y) = EX + EY = \lambda + \mu$ follows by linearity of expectation E (expectation is integration, and integration is linear).

B.

(i) $X = (X_t)$ is a $Ppp(\lambda)$ if for any measurable set A (equivalently, for any interval A), the number $X(A)$ of points of the point process X in A is Poisson distributed with parameter $\lambda|A|$, $X(A) \sim P(\lambda|A|)$, and the numbers of points of X in disjoint sets are independent.

(ii) If $X \sim Ppp(\lambda)$, $Y \sim Ppp(\mu)$, then for any A, $X(A) \sim P(\lambda|A|)$, $Y(A) \sim P(\mu|A|)$, and these are independent as X, Y are independent. So $(X+Y)(A) \sim P((\lambda+\mu)|A|)$, by above. Also, for disjoint sets A, B, $X(A)$, $X(B)$ are independent as X is Poisson, and similarly so are $Y(A)$, $Y(B)$, while both X-counts are independent of both Y-counts as X and Y are independent. Combining, $(X+Y)(A)$ and $(X+Y)(B)$ are independent. This completes the proof that $X+Y$ is $Ppp(\lambda+\mu)$.

(iii) Given that $X+Y$ is a Ppp, its parameter is the constant ν in $E(X+Y)(A) = \nu|A|$. But $E(X+Y)(A) = EX(A) + EY(A) = \lambda|A| + \mu|A|$, so $\nu = \lambda+\mu$.

Q2. As X is a linear combination of independent Gaussian processes X is a Gaussian process (continuous as B_1, B_2 are continuous), and X has zero mean as B_1, B_2 have zero mean. To identify X as standard Brownian motion, it thus suffices to show that its covariance function is $\min(s,t)$, as this is the signature of Brownian motion.

For $s, t \geq 0$,

$$E[X_t X_{t+s}] = (\sigma_1^2 + \sigma_2^2)^{-1}.E[(\sigma_1 B_1(t) + \sigma_2 B_2(t))(\sigma_1 B_1(t+s) + \sigma_2 B_2(t+s))].$$

On the right, replace each $B_i(t+s)$ by $B_i(t) + (B_i(t+s) - B_i(t))$ (the point being that the second term, the increment, is independent of the first, as Brownian motion has independent increments). Then multiply up. This splits the right into the sum of two terms,

$$(\sigma_1^2 + \sigma_2^2)^{-1}.E[(\sigma_1 B_1(t) + \sigma_2 B_2(t))^2]$$

and

$$(\sigma_1^2 + \sigma_2^2)^{-1}.E[(\sigma_1 B_1(t) + \sigma_2 B_2(t))(\sigma_1[B_1(t+s) - B_1(t)] + \sigma_2[B_2(t+s) - B_2(t)])].$$

Multiply out the square in the first term. The cross-term gives 0, as B_1, B_2 are independent and zero-mean: $E(B_1(t)B_2(t)) = EB_1(t).EB_2(t) = 0.0 = 0$. The squared terms give $\sigma_i^2 E[B_i(t)^2] = \sigma_i var(B_i(t) = \sigma_i^2 t$. So the first term is t. The second term is zero, using independence of $B_i(t)$ and $[B_i(t+s) - B_i(t)]$, and of B_1, B_2. Combining, the RHS is t. This is the smaller of the two time-points t, $t+s$ for $s \geq 0$, so in general we get $\min(s,t)$. That is, $cov(X_s, X_t) = \min(s,t)$, and X is standard Brownian motion, as required.

NHB