ltccsoln2.tex
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SOLUTIONS 2

Q1 (Georges BOULIGAND, 1935). *First Proof.* For the region S_1 with area A_1 with base the hypotenuse, side 1: use cartesian coordinates to approximate its area, arbitrarily closely, by decomposing it into small squares of area $dA_1 = dxdy$.

For each such small square on side 1, construct similar small squares on sides 2 and 3, of areas dA_2, dA_3.

By Pythagoras' theorem, $dA_1 = dA_2 + dA_3$.

Summing, we get $A_1 = A_2 + A_3$ arbitrarily closely, and so exactly.

Second Proof. Drop a perpendicular from the right-angled vertex to the hypotenuse. This splits the 'big figure' into two 'smaller figures', each similar to it. With l_1 the length of the hypotenuse and l_2, l_3 those of the other two sides, by similarity lengths scale by l_2/l_1, l_3/l_1 on going from the big figure to the smaller ones, so areas scale by $(l_2/l_1)^2$, $(l_3/l_1)^2$. So $A_2 + A_3 = A_1[(l_2/l_1)^2 + (l_3/l_1)^2] = A_1(l_2^3 + l_3^2)/l_1^2, = A_1$ by Pythagoras' theorem. //

Q2 (*Rejection method*, John von NEUMANN (1903-1957) in 1951).

(i) Suppose we have a density f. Then the area under the curve is 1. The *subgraph* of f is $\{(x, y) : 0 \leq y \leq f(x)\}$. So the area of the subgraph is 1. By definition of density,

$$P(X \in [x, x + dx]) = f(x)dx = dA,$$

where A denotes area under the subgraph to the left of x. So ('probability = area') X has density f iff X is the x-coordinate of a point uniformly distributed over the subgraph of f. So we can go from uniform points (X, Y) on the subgraph to points X with density f by projecting onto the first coordinate; conversely, we can go from such an X to such an (X, Y) by taking $Y \sim Uf(x)$ given $X = x$ (where as usual $U \sim U(0, 1)$).

(ii) If we have a density g that we know how to simulate from, and a density f that we don't know how to simulate from, but

$$f(x) \leq cg(x)$$

for all x and some constant c. We proceed as follows.
1. Simulate from g, i.e. by above
1*. Sample points *uniformly* from the subgraph of g.
2. Stretch the positive y-axis by a factor c.
The points are still uniformly distributed over the subgraph of cg.
3. *Reject* all point not in the subgraph of f (contained in the subgraph of cg, as $f \leq cg$). The remaining points are still uniform, but over the subgraph of f not cg. So:
4. The x-coordinates of the points have density f.

The step that needs checking is 3 – that the non-rejected points are still uniform, but over the subgraph F of f rather than the subgraph G of cg. Before the rejection step, X is uniform over G:

$$X \sim U(G); \qquad P(X \in A) = |A|/|G|, \quad A \subset G$$

(writing $|.|$ for area). Now for $B \subset F$, the distribution of the non-rejected points (i.e. of the points conditional on their being in F) is given by

$$P(X \in B|X \in F) = P(X \in B \& X \in F)/P(X \in F) = P(X \in B \cap F)/P(X \in F)$$

$$= \frac{|B \cap F|}{|G|} / \frac{|F|}{|G|} = |B \cap F|/|F|.$$

This says that the non-rejected points are uniform over F, the subgraph of f, i.e. that they have density f, as required. //