ltccexam2009.tex

LONDON TAUGHT COURSE CENTRE: EXAMINATION, 2009
MEASURE-THEORETIC PROBABILITY

Q1. *Borel-Cantelli Lemmas.*

In what follows, we write 'io' as an abbreviation for 'infinitely often'. For an infinite sequence of events A_n, $n = 1, 2, \ldots$, define the event

$$limsup\ A_n, \quad \text{or} \quad (A_n \quad io), := \bigcap_m \bigcup_{n \geq m} A_n.$$

Show that:
(i) If $\sum P(A_n) < \infty$, then $P(A_n\ io) = 0$.
(ii) If the A_n are independent, and $\sum P(A_n) = \infty$, then $P(A_n\ io) = 1$.

Q2. *Brownian Bridge.* If B is Brownian motion, the process U defined by

$$U_t := B_t - tB_1 \qquad (0 \leq t \leq 1) \qquad (*)$$

is called the *Brownian bridge.*
(i) Show that U is Gaussian, and find its covariance function.
(ii) Show that $U(0) = U(1) = 0$. Why is U called the Brownian bridge?
(iii) Find the wavelet expansion of U (in terms of the Schauder functions Δ_n).
(iv) Is U a martingale with respect to the Brownian filtration $\mathcal{F}_t$?
(v) Give examples of properties of B which U shares, and which U does not share.
(vi) Now extend the range of t in $(*)$ from $[0, 1]$ to $[0, \infty)$. What happens to U_t/t as $t \to \infty$?

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