ltccexamsoln2009.tex

LTCC: SOLUTIONS TO EXAMINATION, 2009
MEASURE-THEORETIC PROBABILITY

Q1(i).

$$\begin{aligned} P(A_n\ io) &= P(\bigcap_m \bigcup_{n=m}^{\infty} A_n) \\ &\leq P(\bigcup_{n=m}^{\infty} A_n) \qquad (P(.)\text{ monotone}) \\ &\leq \sum_{n=m}^{\infty} P(A_n) \qquad (P(.)\text{ subadditive}) \\ &\to 0 \qquad (m \to \infty), \end{aligned}$$

(tail of a convergent series). So

$$P(A_n\ io) = 0.$$

(ii)

$$\begin{aligned} P(\bigcap_m^{\infty} A_n^c) &= \prod_m^{\infty} P(A_n^c) \qquad (\text{independence}) \\ &= \prod_m^{\infty} (1 - P(A_n)). \end{aligned}$$

As $\log(1-x) \leq -x$,

$$\log \prod_m^{\infty} (1 - P(A_n)) = \sum_m^{\infty} \log(1 - P(A_n)) \leq - \sum_m^{\infty} P(A_m) = -\infty,$$

giving

$$P(\bigcap_m^{\infty} A_n^c) = 0 \qquad \text{for each } m,$$

i.e.

$$P((\bigcup_m^{\infty} A_n)^c) = 0, \qquad P(\bigcup_m^{\infty} A_n) = 1 \qquad \text{for each } m.$$

So

$$P(A_n\ io) = P(\bigcap_m \bigcup_m^{\infty} A_n) = 1$$

(union of null sets is null, so on taking complements, intersection of sets of probability 1 has probability 1).

Q2. (i) $B_1 = Z_0$ (see below). So $U_t = B_t - Z_0$ is (from the wavelet expansion for B) a linear combination of the Gaussians Z_n, $n \geq 1$, so is Gaussian. For the mean, $EU_t = E[B_t - tB_1] = E[B_t] - tE[B_1] = 0 - t.0 = 0$. For the covariance,

$$cov(U_s, U_t) = E[U_s U_t] = E[(B_s - sB_1)(B_t - tB_1)]$$

$$= E[B_s B_t] - tE[B_s B_1] - sE[B_1 B_t] + stE[B_1^2] = \min(s,t) - st - st + st :$$

$$cov(U_s, U_t) = \min(s,t) - st.$$

(ii) $U_0 = B_0 - 0.B_1 = 0$, $U_1 = B_1 - 1.B_1 = 0$. U is called the Brownian bridge because it uses a Brownian path to bridge between points $(0,0)$ and $(0,1)$. (U is also called *tied-down Brownian motion* or *pinned Brownian motion.*)

(iii) The wavelet expansion of B is $B_t = \sum_{n=0}^{\infty} l_n Z_n \Delta_n(t)$. Now $\Delta_0(t) = t$, while for $n \geq 1$ Δ_n has support $[k/2^j, (k+1)/2^j]$, where $n = 2^j + k$ $(j, k \geq 0)$. So $\Delta_n(1) = 0$ for $n \geq 1$, and $l_0 = 1$. So putting $t = 1$ gives $B_1 = Z_0$. So the wavelet expansion of U is that of B with the $n = 0$ term dropped:

$$U_t = B_t - tB_1 = \sum_{n=1}^{\infty} l_n Z_n \Delta_n(t).$$

(iv) U is not adapted to the Brownian filtration, as U_t involves B_1, which is not known until Brownian motion reaches time 1, so is not in any of the σ-fields $\mathcal{F}_t$ for $0 \leq t < 1$. So U is not a martingale with respect to the Brownian filtration.

(v) U shares with B the property of having continuous paths, as $U_t = B_t - tB_1$ and both terms on the right are continuous in t, hence so is the left. U does not share the Brownian scaling property: it is tied to the point $t = 1$, which is fixed in time and so does not scale.

(vi) If the time-set in $(*)$ is now extended from $[0,1]$ to $[0,\infty)$:

$$U_t/t = B_t/t - B_1.$$

Now by Brownian scaling $B_t/\sqrt{t}$ has the same distribution as B_1, so is a standard normal random variable, Z say, and so B_t/t is distributed as $Z/\sqrt{t} \to 0$ as $t \to \infty$. So

$$B_t/t \to 0, \qquad U_t/t \to -B_1 \qquad (t \to \infty),$$

the convergence of B_t/t being in the sense of having the same distribution as something tending to 0. (The convergence is actually with probability 1, by the strong law of large numbers. The above gives convergence in distribution to a constant, which is equivalent to convergence in probability to this constant. Any sense of convergence will do here.)

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