London Taught Course Centre: Measure-Theoretic Probability

Solutions to Examination, 2010

Q1. (i)

$$\begin{aligned}\psi(t) = E[e^{itY}] &= E[\exp\{it(X_1 + \ldots + X_N)\}] \\ &= \sum_n E[\exp\{it(X_1 + \ldots + X_N)\}|N = n].P(N = n) \\ &= \sum_n e^{-\lambda}\lambda^n/n!.E[\exp\{it(X_1 + \ldots + X_n)\}] \\ &= \sum_n e^{-\lambda}\lambda^n/n!.(E[\exp\{itX_1\}])^n \\ &= \sum_n e^{-\lambda}\lambda^n/n!.\phi(t)^n \\ &= \exp\{-\lambda(1 - \phi(t))\}.\end{aligned}$$

Differentiate:

$$\psi'(t) = \psi(t).\lambda\phi'(t),$$

$$\psi''(t) = \psi'(t).\lambda\phi'(t) + \psi(t).\lambda\phi''(t).$$

As $\phi(t) = E[e^{itX}]$, $\phi'(t) = E[iXe^{itX}]$, $\phi''(t) = E[-X^2e^{itX}]$. So ($\phi(0) = 1$ and) $\phi'(0) = i\mu$, $\phi''(0) = -E[X^2]$,

$$\psi'(0) = \lambda\phi'(0) = \lambda.i\mu,$$

and as also $\psi'(0) = iEY$, this gives $EY = \lambda\mu$. Similarly,

$$\psi''(0) = i\lambda\mu.i\lambda\mu + \lambda\phi''(0) = -\lambda^2\mu^2 - \lambda E[X^2],$$

and also ($\psi(0) = 1$, $\psi'(0) = i\lambda\mu$ and) $\psi''(0) = -E[Y^2]$. So

$$var\ Y = E[Y^2] - [EY]^2 = \lambda^2\mu^2 + \lambda E[X^2] - \lambda^2\mu^2 = \lambda E[X^2].$$

(ii) Given N, $Y = X_1 + \ldots + X_N$ has mean $NEX = N\mu$ and variance $N\ var\ X = N\sigma^2$. As N is Poisson with parameter λ, N has mean λ and variance λ. So by the Conditional Mean Formula,

$$EY = E[E(Y|N)] = E[N\mu] = \lambda\mu.$$

By the Conditional Variance Formula,

$$var\ Y = E[var(Y|N)] + var\ E[Y|N] = E[Nvar\ X] + var[N\ EX]$$

$$= EN.var\ X + var\ N.(EX)^2 = \lambda[E(X^2) - (EX)^2] + \lambda.(EX)^2 = \lambda E[X^2].$$

Q2. (i) For $t \neq 0$, X is Gaussian with zero mean (as B is), and continuous (again, as B is). The covariance of B is $\min(s,t)$. The covariance of X is

$$\begin{aligned} cov(X_s, X_t) &= cov(sB(1/s), tB(1/t)) \\ &= E[sB(1/s).tB(1/t)] \\ &= st.E[B(1/s)B(1/t)] \\ &= st.cov(B(1/s), B(1/t)) \\ &= st.\min(1/s, 1/t) = \min(t,s) = \min(s,t). \end{aligned}$$

This is the same covariance as Brownian motion. So, away from the origin, X *is* Brownian motion, as a Gaussian process is uniquely characterized by its mean and covariance (from the properties of the multivariate normal distribution). So X is continuous. So we can define it at the origin by continuity. So X is Brownian motion everywhere – X is BM.

(ii) Since Brownian motion is 0 at the origin, $X(0) = 0$. Since Brownian motion is continuous at the origin, $X(t) \to 0$ as $t \to 0$. This says that

$$tB(1/t) \to 0 \qquad (t \to 0),$$

which is

$$B(t)/t \to 0 \qquad (t \to \infty).$$

Note. For t integer, this is the Strong Law of Large Numbers applied to the distribution of $B(1)$, which is standard normal. The above neat proof by *time-inversion* follows from the proof of existence of Brownian motion (defined to be continuous), given in lectures by the wavelet expansion.

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