VERY SLOWLY VARYING FUNCTIONS – II
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Abstract. This paper is a sequel to both Ash, Erdös and Rubel [AER], on very slowly varying functions, and [BOst1], on foundations of regular variation. We show that generalizations of the Ash-Erdős-Rubel approach – imposing growth restrictions on the function h, rather than regularity conditions such as measurability or the Baire property – lead naturally to the main result of regular variation, the Uniform Convergence Theorem.

1 Introduction and Main Result

We work with the Karamata theory of regular and slow variation; see [BGT] – BGT in what follows – for a monograph account. Here the main result is the Uniform Convergence Theorem – UCT below – which asserts that the defining pointwise convergence for slow variation in fact holds uniformly on compact sets if the function h in question is either (Lebesgue) measurable, or has the Baire property, but not in general. The outstanding foundational question of the theory – raised and left open in [BG1], [BG2], BGT – is what common generalization of measurability and the Baire property suffices. This question is answered in [BOst1], where we obtain two sets of conditions, each necessary and sufficient. Our results are of two kinds. The first uses 'naive set theory' and is thus immediately accessible to analysts and probabilists, for whom regular variation is such a necessary and useful working tool. The second makes use of the 'heavy machinery' of descriptive set theory, and so is perhaps more easily accessible to mathematical logicians.

A very few papers in regular variation are able to make progress without imposing regularity conditions. Foremost among these are the Ash-Erdős-Rubel paper [AER], where a growth condition is used instead, and the work of Heiberg [Hei] and Seneta [Sen1], [Sen2], where side-conditions involving the limsup are imposed instead. Informed by the viewpoint of [BOst1], we generalize the results of these papers, but following only the ordinary or 'naive' set theory approach – that is, without use of descriptive set theory. Our results are thus immediately accessible to the user communities of analysts and probabilists, granted only an acquaintance with BGT, the standard work on the subject.

We will apply the Main Theorem UCT of [BOst1] to derive a new, simple, necessary and sufficient condition on a function h so that it obeys the UCT. In

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Section 2 we show how the simple conditions may be usefully relaxed and then use the latter conditions to identify why the example of [AER] does not satisfy the UCT.

We begin by defining the key notions of the theory of regular variation. Then we recall the definitions and two theorem of [BOst1] which we will need here. The theory is concerned with the consequences of a relationship of the form

$$f(\lambda x)/f(x) \to g(\lambda) \qquad (x \to \infty) \qquad \forall \lambda > 0, \tag{RV}$$

for functions defined on $\mathbb{R}_+$. The limit function g must satisfy the Cauchy functional equation

$$g(\lambda\mu) = g(\lambda)g(\mu) \qquad \forall \lambda, \mu > 0. \tag{CFE}$$

Subject to a mild regularity condition, (CFE) forces g to be a power:

$$g(\lambda) = \lambda^\rho \qquad \forall \rho > 0. \tag{ρ}$$

Then f is said to be *regularly varying* with *index* ρ, written $f \in R_\rho$.

The case $\rho = 0$ is basic. A function f is called *slowly varying*, if $f \in R_0$, i.e. if

$$f(\lambda x)/f(x) \to 1 \qquad (x \to \infty) \qquad \forall \lambda > 0. \tag{SV}$$

Slowly varying functions are often written ℓ (for *lente*, or *langsam*). The basic theorem of the subject is the Uniform Convergence Theorem (UCT), which states that under appropriate assumptions if (SV) holds, then the convergence is *uniform* on compact sets of λ values in $(0, \infty)$. Necessary and sufficient assumptions for UCT have only recently been given (in [BOst1]) and are quoted below for convenience. While regular variation is usually used in the multiplicative formulation above, for proofs in the subject it is usually more convenient to use an additive formulation. Writing $h(x) := \log f(e^x)$ (or $\log \ell(e^x)$ as the case may be), the relation above becomes

$$h(x+u) - h(x) \to 0 \qquad (x \to \infty) \qquad \forall u \in \mathbb{R}. \tag{SV_+}$$

Here the functions are defined on $\mathbb{R}$, whereas in the multiplicative notation functions are defined on $\mathbb{R}_+$. We find it helpful to use the notation $h_x(u) = h(x+u) - h(x)$.

Definitions.

(i) The **ε-level set** (of h_x) is defined to be the set

$$H^\varepsilon(x) = \{t : |h(t+x) - h(x)| < \varepsilon\}.$$

(ii) For $\mathbf{x} = \{x_n : n \in \omega\}$ an arbitrary sequence tending to infinity, the **x-stabilized ε-level set** (of h) is defined to be the set

$$T_k^\varepsilon(\mathbf{x}) = \textstyle\bigcap_{n=k}^\infty H^\varepsilon(x_n) \text{ for } k \in \omega.$$

Here ω denotes the set of natural numbers $0, 1, 2, \ldots$. Note that

$$T_0^\varepsilon(\mathbf{x}) \subseteq T_1^\varepsilon(\mathbf{x}) \subseteq T_2^\varepsilon(\mathbf{x}) \subseteq \ldots \text{ and } T_k^\varepsilon(\mathbf{x}) \subseteq T_k^\eta(\mathbf{x}) \text{ whenever } \varepsilon < \eta. \tag{1}$$

If h is slowly varying, then $\mathbb{R} = \bigcup_{k\in\omega} T_k^\varepsilon(\mathbf{x})$.

(iii) The basic **No Trumps** combinatorial principle (there are several), denoted $\mathbf{NT}(\{T_k : k \in \omega\})$, refers to a family of subsets of reals $\{T_k : k \in \omega\}$ and means the following.

For every bounded sequence of reals $\{u_m : m \in \omega\}$ there are $k \in \omega$, $t \in \mathbb{R}$ and an infinite set $\mathbb{M} \subseteq \omega$ such that

$$u_m + t \in T_k \text{ for all } m \text{ in } \mathbb{M}.$$

In words: the translate of some *subsequence* of $\{u_m\}$ is contained in some T_k. We will also say that $\{T_k : k \in \omega\}$ **traps sequences by translation.**

Main Theorem (UCT). *For h slowly varying, the following are equivalent.*
(i) *The UCT holds for h.*
(ii) *The principle $\mathbf{1\text{-}NT}_h$ holds: for every $\varepsilon > 0$ and every sequence $\mathbf{x}$ tending to infinity, the stabilized ε-level sets $\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\}$ of h trap bounded sequences by translation.*

In loose notation:$(\forall\varepsilon > 0)(\forall\mathbf{x})$ $\mathbf{NT}(\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\})$.
(iii) *For every $\varepsilon > 0$ and for every sequence $\mathbf{x}$ tending to infinity, the stabilized ε-level sets $\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\}$ of h contain all the bounded sequences.*

The property in (iii) is called the full-inclusion or F-analogue of $\mathbf{1\text{-}NT}_h$. For the proof see [BOst1], where it is also shown that either of the conditions (ii) or (iii) holds for measurable h, and also for h with the Baire property. We will also need the following result from [BOst1].

Theorem (Bounded Equivalence Principle). *For h a slowly varying function the following are equivalent.*

(i) *The family $\{T_n^\varepsilon(\mathbf{x}) : n \in \omega\}$ traps bounded sequences for any sequence $\mathbf{x}$ tending to infinity, and any positive ε.*

(ii) *Whenever $\{u_n\}$ is a bounded sequence, and $\{x_n\}$ tends to infinity*

$$\lim_{n\to\infty} |h(u_n + x_n) - h(x_n)| = 0. \tag{2}$$

(iii) *For any sequence $\mathbf{x}$ tending to infinity, and any positive ε, the family $\{T_n^\varepsilon(\mathbf{x}) : n \in \omega\}$ ultimately contains almost all of any bounded sequence $\{u_n\}$. That is, for any bounded sequence $\{u_n\}$ there is k such that*

$$\{u_m : m > k\} \subseteq T_n^\varepsilon(\mathbf{x}) \text{ for all } n > k. \tag{3}$$

(iv) *The UCT holds for h.*

Definition. We say that h satisfies the **Heiberg-Lipschitz condition** if there are two positive functions φ, g defined on $\mathbb{R}_+$ such that:
(i) $g(x)$ is decreasing to 0 as $x \to \infty$;
(ii) $\varphi(t) \to \infty$ as $t \to \infty$;

(iii) for all $x, t > 0$, there is $x(t)$ between x and $x + t$ such that

$$|h(t+x) - h(x)| \; = \varphi(t)g(x(t)). \tag{4}$$

The final condition is modelled after the mean-value theorem. Note that the assumptions imply that for all $x, t > 0$

$$\varphi(t)g(x+t) \leq |h(t+x) - h(x)| \; \leq \varphi(t)g(x).$$

This is the information which makes the proof of our main theorem transparent; we show later how to relax these assumptions to obtain a more useful formulation of the basic paradigm. When studying slowly varying functions h in the context of the Uniform Convergence Theorem (UCT) it helps to paraphrase the concepts by reference to the notation introduced earlier:

$$h_x(u) = h(u+x) - h(x).$$

Regarding x as a parameter and $h_x(u)$ as an 'approximately-additive' function of u, slow variation is just pointwise convergence to zero of the family $\{h_x\}$ as $x \to \infty$ (at all single points u). Thus UCT is the qualified assertion that pointwise convergence of the family $\{h_x\}$ implies uniform convergence over compact sets of u. In this language, the simple Heiberg-Seneta condition 4 'factorizes out of h_x its dependence on x' locally. The original (i.e. First – see below) Heiberg-Seneta Theorem factorizes out 'dependence on x at infinity', studying in essence an appropriate application of L'Hospital's Rule. Our Generalized Heiberg-Seneta Theorem of Section 2 is then the 'direct comparison' analogue.

Remark. The preceding definition subsumes the case of any increasing, differentiable concave function $h(x)$ satisfying the celebrated 'Inada conditions' of Economic Theory, introduced in [Inada]. This class includes $\log x$ and the power functions x^ρ with $0 < \rho < 1$. Indeed, for h satisfying the Heiberg-Lipschitz condition, we have, for $t > 0$, for some x^* with $x < x^* < x + t$, that

$$0 < h(x+t) - h(x) = h'(x^*)t.$$

Thus with $g(x) = h'(x)$ and $x(t) = x^*$ the conditions are met since $g(x)$ is decreasing to 0 as $x \to \infty$.

Observation. *If h satisfies the Heiberg-Lipschitz condition, then h is slowly varying.*

For,

$$\lim_{x\to\infty} |h(t+x) - h(x)| \leq \lim_{x\to\infty} \varphi(t)g(x) = 0.$$

Our main result follows (for details of the *First Heiberg-Seneta Theorem* see the closing discussion in Section 3). This new theorem thus complements [Hei], [Sen1], [Sen2], cf. BGT Theorem 1.4.3 p. 18-19.

Theorem (Second Heiberg-Seneta Theorem). *For h satisfying the Heiberg -Lipschitz condition the following are equivalent.*

(i) *UCT holds for h.*

(ii) *The family $\{\varphi^{-1}((0,n)) : n \in \omega\}$ traps sequences by translation.*

(iii) *The family $\{\varphi^{-1}((0,n)) : n \in \omega\}$ contains almost all terms of every bounded sequence.*

(iv) *The family $\{\varphi^{-1}((0,n)) : n \in \omega\}$ contains every bounded sequence.*

Proof. We will show first (a) that (ii) implies (i), and then (b) that (i) implies (ii).

Clearly (iv) implies (iii) and (iii) implies (ii). The proof will thus be complete when in (c) we explain how to adapt the notation used in the proof of (b) so that it reads as a proof of (i) implies (iv).

(a) Proof that (ii) implies (i).

Let $\mathbf{x} = \{x_n\}$ be any sequence tending to infinity, let $\mathbf{u} = \{u_m\}$ be any bounded sequence and suppose that the condition of the Bounded Equivalence Principle, namely

$$\lim_{n\to\infty} |h(u_n + x_n) - h(x_n)| = 0,$$

fails. Thus we suppose that for some $\varepsilon > 0$ and for $n = 1, 2, ...$ we have

$$|h(x_n + u_n) - h(x_n)| > 2\varepsilon. \tag{5}$$

Working by analogy with ε-level sets, define the reduced level sets by

$$H_n^- = \{t : \varphi(t) < \varepsilon/g(x_n)\}.$$

Thus

$$H_n^- \subseteq \{y : |h(x_n + y) - h(x_n)| < \varepsilon\} = H^\varepsilon(x_n).$$

Observe next that

$$\begin{aligned} H_n^- - u_m &= \{y : (\exists t)[y = t - u_m \ \&\ t \in H_n^-]\} = \{y : (\exists t)[t = u_m + y \ \&\ \varphi(t) < \varepsilon/g(x_n)]\} \\ &= \{y : \varphi(u_m + y) < \varepsilon/g(x_n)\}. \end{aligned}$$

Since $\{\varphi^{-1}((0,n)) : n \in \omega\}$ is sequence trapping, there are N, y and infinite $\mathbb{M}$ such that

$$\{u_m + y : m \in \mathbb{M}\} \subseteq \varphi^{-1}((0,N)), \text{ i.e. } \{\varphi(u_m + y) : m \in \mathbb{M}\} \subseteq (0,N).$$

But, for some k large enough, we have $\varepsilon/g(x_k) > N$. Hence, for this y, we have, for $n \geq k$, that

$$\{\varphi(u_m + y) : m \in \mathbb{M}\} \subseteq (0,N) \subseteq (0, \varepsilon/g(x_n)).$$

Thus by definition of H_n we have, for all $m \in \mathbb{M}$, that

$$y \in \cap_{n=k}^\infty H_n^- - u_m.$$

We now claim that, for any $n \geq k$ with $n \in \mathbb{M}$, we have

$$|h(x_n + u_n) - h(x_n + u_n + y)| \geq \varepsilon. \tag{6}$$

Indeed, we would otherwise have, for any such n, that

$$|h(x_n + u_n) - h(x_n + u_n + y)| < \varepsilon. \tag{7}$$

But referring to $x = x_n$ and $t = u_n + y$ in clause (iii) of the Heiberg-Lipschitz condition we have, since $y \in H_n^- - u_n$, that

$$|h(u_n + y + x_n) - h(x_n)| \leq \varphi(u_n + y)g(x_n) < \varepsilon,$$

and this combined with (7) yields

$$|h(x_n + u_n) - h(x_n)| < 2\varepsilon,$$

a contradiction to our standing assumption (5).

Define $v_n = x_n + u_n$ (which tends to infinity). Then the relation (6) yields that

$$|h(v_n + y) - h(v_n)| \geq \varepsilon,$$

for infinitely many n, which contradicts that h is slowly varying. $\square$ (a)

(b) Proof that (i) implies (ii).

Let $\mathbf{x} = \{x_n\}$ be any sequence tending to infinity and let $\mathbf{u} = \{u_m\}$ be any positive bounded sequence (otherwise pass to a subsequence). Assume for some $b > 0$ that for all $m \in \omega$ we have

$$0 \leq u_m \leq b.$$

Again working by analogy with ε-level sets, define the expanded level sets by

$$H_n^+ = \{t : \varphi(t) < \varepsilon / g(x_n + t)\}.$$

Thus

$$H^\varepsilon(x_n) = \{y : |h(x_n + y) - h(x_n)| < \varepsilon\} \subseteq H_n^+,$$

since, for $t \in H^\varepsilon(x_n)$,

$$\varphi(t)g(x_n + t) \leq |h(t + x_n) - h(x_n)| < \varepsilon.$$

Now if UCT holds, then by the Main Theorem of [BOst1] $\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\}$ traps sequences, so for some y, infinite $\mathbb{M}$ and $k \in \omega$, we have

$$\{y + u_m : m \in \mathbb{M}\} \subseteq T_k^\varepsilon(\mathbf{x}) = \bigcap_{n=k}^{\infty} H^\varepsilon(x_n) \subseteq H^\varepsilon(x_k) \subseteq H_k^+ = \{t : \varphi(t) < \varepsilon / g(x_k + t)\},$$

i.e.

$$\{y + u_m : m \in \mathbb{M}\} \subset \{t : \varphi(t) < \varepsilon/g(x_k + t)\}.$$

Thus we have, for $m \in \mathbb{M}$, that

$$\varphi(y + u_m) < \varepsilon/g(x_k + y + u_m) \leq \varepsilon/g(x_k + y + b).$$

Choose an integer N such that, for all $t > N$, we have $\varphi(t) > \varepsilon/g(x_k+y+b)$. Then we have

$$\{y + u_m : m \in \mathbb{M}\} \subset \varphi^{-1}(0, N),$$

as required. □ (b)

(c) Modification to (b). Given the Main Theorem UCT of [BOst1] we may clearly adapt the proof just given in (b) to show that (i) implies (iv) by putting $y = 0$ and $\mathbb{M} = \omega$. □ (c)

2 A generalization

In this section we show one possible way to move away from the context dictated by the mean-value theorem and still have a corresponding Second Heiberg-Seneta Theorem. Some further alternative formulations are discussed in Section 3.

Definition. We say that h satisfies the **generalized Heiberg-Lipschitz condition** if

(a) there is a function φ defined on $\mathbb{R}_+$ such that: $\varphi(t) \to \infty$ as $t \to \infty$;

(b) there are functions g_+, g_- defined on $\mathbb{R}_+^2$ such that, for $x, t > 0$, we have

$$g_-(x, \varphi(t)) \leq |h(x + t) - h(x)| \leq g_+(x, \varphi(t)), \tag{8}$$

(c) and, for all $\varepsilon > 0$ small enough, the solution sets of

$$g_\pm(x, y) < \varepsilon$$

are bounded and, for some functions $\psi_\pm(x, \varepsilon)$, take the form $\{y : y < \psi_\pm(x, \varepsilon)\}$;

(d) $\lim_{x\to\infty} \psi_+(x, \varepsilon) = \infty$.

Observation. *If h satisfies the generalized Heiberg-Lipschitz condition, then h is slowly varying.*

Indeed, given t, $\varepsilon > 0$ there exists $X > 0$, by condition (d), such that $\psi_+(x, \varepsilon) > \phi(t)$ for $x > X$, or equivalently, from the condition (c), such that $g_+(x, \varphi(t)) < \varepsilon$ for $x \geq X$. In this case we conclude, for $x > X$, that

$$|h(x + t) - h(x)| < \varepsilon.$$

Note that the observation relies only on the right-hand inequality in (8).

Theorem (Generalized Heiberg-Seneta Theorem). *For h satisfying the generalized Heiberg-Lipschitz condition the following are equivalent.*

(i) *UCT holds for h.*

(ii) *The family $\{\varphi^{-1}((0,n)) : n \in \omega\}$ traps sequences by translation.*

(iii) *The family $\{\varphi^{-1}((0,n)) : n \in \omega\}$ contains almost all terms of every bounded sequence.*

(iv) *The family $\{\varphi^{-1}((0,n)) : n \in \omega\}$ contains every bounded sequence.*

Proof. We follow the proof structure of the Second Heiberg-Seneta Theorem.

(a) Proof of UCT from (ii) . As before, suppose for some $\varepsilon > 0$ and for $n = 1, 2, ..$ that we have

$$|h(x_n + u_n) - h(x_n)| \geq 2\varepsilon. \tag{9}$$

As expected, put

$$H_n^- = \{t : g_+(x_n, \varphi(t)) < \varepsilon\}.$$

Thus

$$H_n^- = \{y : |h(y + x_n) - h(x_n)| < \varepsilon\} \subseteq H^\varepsilon(x_n),$$

since

$$|h(y + x_n) - h(x_n)| \leq g_+(x_n, \varphi(y)) < \varepsilon.$$

As before,

$$H_n^- - u_m = \{y : g_+(x_n, \varphi(u_m + y)) < \varepsilon\} \subseteq \{y : \varphi(y + u_m) < \psi_+(x_n, \varepsilon)\}.$$

Since $\{\varphi^{-1}((0,n)) : n \in \omega\}$ is sequence trapping, there are N, y and infinite $\mathbb{M}$ such that

$$\{u_m + y : m \in \mathbb{M}\} \subseteq \varphi^{-1}((0,N)), \text{ i.e. } \{\varphi(u_m + y) : m \in \mathbb{M}\} \subseteq (0,N).$$

Since $\lim_{x\to\infty} \psi_+(x,\varepsilon) = \infty$, for some k large enough, we have $\psi_+(x_n,\varepsilon) > N$ for all $n \geq k$. Hence, for this y we have, for $n \geq k$, that

$$\{\varphi(u_m + y) : m \in \mathbb{M}\} \subseteq (0,N) \subseteq (0, \psi_+(x_n,\varepsilon)).$$

Thus by definition of H_n^- we have, for all $m \in \mathbb{M}$, that

$$y \in \cap_{n=k}^{\infty} H_n^- - u_m.$$

We now claim that, for any $n \geq k$ with $n \in \mathbb{M}$, we have

$$|h(x_n + u_n) - h(x_n + u_n + y)| \geq \varepsilon. \tag{10}$$

Indeed, we would otherwise have for any such n that

$$|h(x_n + u_n) - h(x_n + u_n + y)| < \varepsilon. \tag{11}$$

But referring to $x = x_n$ and $t = u_n + y$ in clause (b) of the generalized Heiberg-Lipschitz condition we have, since $y \in H_n^- - u_n$, that

$$|h(u_n + y + x_n) - h(x_n)| \leq g_+(x_n, \varphi(y + u_n)) < \varepsilon,$$

and this combined with (11) yields

$$|h(x_n + u_n) - h(x_n)| < 2\varepsilon,$$

a contradiction to our standing assumption (9).

Define $v_n = x_n + u_n$ (which tends to infinity); then the relation (10) yields that

$$|h(v_n + y) - h(v_n)| \geq \varepsilon,$$

for infinitely many n, which contradicts that h is slowly varying. □ (a)

(b) Proof that UCT implies condition (ii). As expected put

$$H_n^+ = \{t : g_-(x_n, \varphi(t)) < \varepsilon\}.$$

Thus

$$H^\varepsilon(x_n) = \{y : |h(y + x_n) - h(x_n)| < \varepsilon\} \subseteq H_n^+,$$

since, for $y \in H^\varepsilon(x_n)$,

$$g_-(x_n, \varphi(y)) \leq |h(y + x_n) - h(x_n)| < \varepsilon.$$

Now if UCT holds then, by the No Trumps Theorem of [BOst1], $\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\}$ traps sequences, so for some y, infinite $\mathbb{M}$, and k we have, as before, that

$$\{y + u_m : m \in \mathbb{M}\} \subseteq T_k^\varepsilon(\mathbf{x}) \subseteq H^\varepsilon(x_k) \subseteq H_k^+ = \{t : g_-(x_k, \varphi(t)) < \varepsilon\},$$

i.e.

$$\{y + u_m : m \in \mathbb{M}\} \subset \{t : \varphi(t) < \psi_-(x_k, \varepsilon)\}.$$

Thus we have, for $m \in \mathbb{M}$, that

$$\varphi(y + u_m) < \psi_-(x_k, \varepsilon).$$

Choose an integer N such that, for all $t > N$, we have $\varphi(t) > \psi_-(x_k, \varepsilon)$. Then we have

$$\{y + u_m : m \in \mathbb{M}\} \subseteq \varphi^{-1}(0, N),$$

as required. $\square$ (b)

(c) Modifications to (b). Now if UCT holds then, by part (iii) of the Main Theorem of [BOst1] (see Section 1 above), $\{T_k^\varepsilon(\mathbf{x}) : k \in \omega\}$ contains all sequences, so the proof in (b) may be re-read with $y = 0$ and $\mathbb{M} = \omega$. $\square$ (c)

We now take the view that $\mathbb{R}$ is a vector space over the field $\mathbb{Q}$. For the purposes of the next result, we need to assume the existence of a (Hamel) basis in this vector space. Its existence is assured by the Axiom of Choice (AC); as is well-known, (AC) implies that every vector space has a basis. We note in passing the converse is also true; see [Bl].

Fix a Hamel basis H which includes 1. Let $n(t)$ be the cardinality of the smallest subset of H which spans t (over $\mathbb{Q}$). We now use the last theorem to explain why the following slowly varying function, introduced in [AER], does not obey UCT. Whilst our proof is slightly longer than that in BGT p. 10-11, we feel that it casts rather more light on what is happening.

Proposition. *The slowly varying function* $h(x) = \log(x + n(x))$ *does not satisfy UCT.*

Proof. We begin by establishing the left inequality (8) for all rational x and the right inequality for all x. (The latter implies that h is slowly varying.)

Applying the mean-value theorem to the logarithm function, we have for $h(x) = \log(x + n(x))$ that

$$\frac{t + n(x+t) - n(x)}{x + t + n(x+t)} \leq |h(t+x) - h(x)| \leq \frac{t + n(x+t) - n(x)}{x} \leq \frac{t + n(t)}{x},$$

since $n(x+t) \leq n(x) + n(t)$. Now for $x \in \mathbb{Q}$ we have $n(x) = 1$ and so

$$n(t) \leq n(x+t) \leq n(t) + 1.$$

Thus putting $\varphi(t) = t + n(t)$ and noting that $\varphi(t)$ tends to infinity we have for $x \in \mathbb{Q}$

$$\frac{\varphi(t) - 1}{x + \varphi(t) + 1} = \frac{t + n(t) - 1}{x + t + n(t) + 1} \leq |h(t+x) - h(x)| \leq \frac{\varphi(t) + 1}{x}.$$

Putting

$$g_-(x, \varphi(t)) := \frac{\varphi(t) - 1}{x + \varphi(t) + 1}, \qquad g_+(x, \varphi(t)) := \frac{\varphi(t) + 1}{x},$$

we therefore have

$$g_-(x, \varphi(t)) \leq |h(t+x) - h(x)| \ (x \in \mathbb{Q}_+) \text{ and } |h(t+x) - h(x)| \leq g_+(x, \varphi(t)) \ (x \in \mathbb{R}_+).$$

Let $0 < \varepsilon < 1$. The solution set $g_-(x, y) \leq \varepsilon$ is bounded by the line

$$y = \psi_-(x, \varepsilon) = \frac{\varepsilon(x+1) + 1}{1 - \varepsilon}.$$

With (8) established for $x \in \mathbb{Q}$, we may now apply the general theorem to show that UCT fails. This we may do by restricting attention to any sequence of rationals $\{x_n\}$ that tends to infinity. By the Main Theorem in [BOst1] all we need do is check that the family of sets $T_k = \{t : \varphi(t) \leq k\}$ is not sequence trapping. Indeed choose t_m in $[0, 1]$ so that $n(t_m) = m$. By passing to a subsequence we may, without loss of generality, assume that t_m converges. But for any y and any infinite $\mathbb{M}$ the subsequence $\varphi(t_m + y)$ for m in $\mathbb{M}$ is unbounded, since $t_m + y + n(t_m) \leq \varphi(t_m + y)$. Hence $\{t_m + y : m \in \mathbb{M}\}$ is not trapped by T_k for any k. $\square$

3 Complements

De Haan theory. The study of functional relations of the form (RV), or (RV_+), is *Karamata theory*, in the terminology of BGT Ch. 1,2. Related is the study of *de Haan theory* – that of relations of the form

$$\frac{f(\lambda x) - f(x)}{g(x)} \to h(\lambda) \qquad (x \to \infty) \qquad \forall \lambda > 0 \qquad (deH)$$

(BGT, Ch. 3). See BGT §3.0 for the inter-relationships between the two (de Haan theory both contains Karamata theory, and refines it by filling in 'gaps'). Our approach here to Karamata theory extends to de Haan theory along similar lines.

In de Haan theory, the relevant limit function in (deH) is

$$h(\lambda) = \begin{cases} \frac{\lambda^\rho - 1}{\rho}, & \rho \neq 0, \\ \log \lambda, & \rho = 0. \end{cases}$$

The Ash-Erdös-Rubel results [AER] and Heiberg-Lipschitz condition have something of a de Haan rather than a Karamata character. See e.g. BGT Th. 3.1.10a,c for illustrations of this.

Weakening quantifiers. It is both interesting and useful to see to what extent the quantifier $\forall$ in (RV), (deH) may be weakened to 'for some', plus some side-condition. The prototypical result here is (BGT Th. 1.4.3 in the Karamata case – cf. Th. 3.2.5 in the de Haan case) the following result.

Theorem (First Heiberg-Seneta Theorem). *Write*

$$g^*(\lambda) := \limsup_{x \to \infty} f(\lambda x)/f(x),$$

and assume that

$$\limsup_{\lambda \downarrow 1} g^*(\lambda) \leq 1.$$

Then for a positive function f, the following are equivalent:
(i)(RV) and (ρ) hold for some ρ.

(ii) The limit $g(\lambda)$ in (RV) exists for all λ in a set of positive measure, or a non-meagre Baire set.
(iii) The limit $g(\lambda)$ in (RV) exists, finite, for all λ in a dense subset of $(0, \infty)$.
(iv) The limit $g(\lambda)$ in (RV) exists, finite, for $\lambda = \lambda_1, \lambda_2$ with $(\log \lambda_1)/(\log \lambda_2)$ finite and irrational.

This question of weakening of quantifiers is treated in detail in [BG1] (where the above is Th. 5.7). The original motivation was the study of Frullani integrals; see [BG2] §6, BGT §1.6.4, Berndt [Ber], p. 466-467.

Further generalizations. We note that the lower bound may be taken in the form

$$g_-(x + \varphi(t))\varphi(t),$$

provided that for all $\varepsilon > 0$ small enough, the solution set of

$$g_-(x + y)y < \varepsilon$$

is bounded and takes the form $\{y : y < \psi_-(x, \varepsilon)\}$. Rewriting the solution set as

$$S(x, \varepsilon) = \left\{ y : 0 \leq y < G(y) = \frac{\varepsilon}{g_-(x + y)} \right\},$$

we see that $0 \in S$. Thus $\psi_-(x, \varepsilon)$ is well-defined iff $\sup S(x, \varepsilon) < \infty$. Geometrically, the assumption requires the graph of $G(y)$ to cross the ray of slope 1 from the origin once so as to be to be ultimately below it. The condition is satisfied in the quoted example of [AER]. Putting $\varphi(t) = t - 1 + n(t)$, again a function tending to infinity, we have that for $x \in \mathbb{Q}$

$$\frac{\varphi(t)}{x + \varphi(t) + 2} = \frac{t + n(t) - 1}{x + t + n(t) + 1} \leq |h(t + x) - h(x)|.$$

Let $0 < \varepsilon < 1$. The required solution set is thus bounded by the line

$$\psi_-(x, \varepsilon) = \frac{\varepsilon(x + 2)}{1 - \varepsilon},$$

with slope less than unity.

One can introduce other conditions relaxing the location of the term $x(t)$ of the simple Heiberg-Lipschitz condition (4), say by bounding $|h(t + x) - h(x)|$ above and below 'functionally', i.e. in terms of functions of x and functions of t, so long as one can recover corresponding finite functions $\psi_\pm(x, \varepsilon)$ with $\lim_{x \to \infty} \psi_+(x, \varepsilon) = \infty$.

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