Cauchy's functional equation and extensions: Goldie's equation and inequality, the Gołąb-Schinzel equation and Beurling's equation

by

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To Ranko Bojanić on his 90^{th} birthday.

Abstract. The Cauchy functional equation is not only the most important single functional equation, it is also central to regular variation. Classical Karamata regular variation involves a functional equation and inequality due to Goldie; we study this, and its counterpart in Beurling regular variation, together with the related Gołąb-Schinzel equation.
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1 Introduction

We are concerned with the most important single functional equation, the *Cauchy Functional Equation* (CFE)

$$K(x+y) = K(x) + K(y), \qquad k(xy) = k(x)k(y), \qquad (CFE)$$

to give both the additive and multiplicative versions. For background, see the standard work by Kuczma [Kuc]. This is known to be crucial to the theory of regular variation, in both its Karamata form (see Ch. 1 of [BinGT], BGT below) and its Bojanić-Karamata/de Haan form (BGT Ch. 3, [BojK]). A close study of these involves a certain functional equation ([BinG], BGT), which we call here the *Goldie* functional equation ((GFE) – see below) [1]. One of the themes of Kuczma's book is the interplay between functional equations and inequalities; he focusses particularly on the Cauchy functional

[1] The equation occurs first in joint work by the first author and Goldie; the first author is happy to confirm that the argument is in fact due to Goldie, whence the name.

equation and Jensen's inequality. Even more closely linked to (CFE) is the functional inequality of *subadditivity*, and this has its counterpart in *Goldie's functional inequality* ((GFI) – see below).

Closely related to the Karamata theory of slow and regular variation is the theory of *Beurling slow and regular variation*. This has an odd history. Beurling (who was a perfectionist) never published it (it is not mentioned in the two volumes of his Collected Works). He introduced it in lectures in the 1940s for use in his *Tauberian theorem*. Beurling's Tauberian theorem appeared in the 1972 papers of Moh [Moh] and Peterson [Pet]; it was used by the first author [Bin] in 1981 in probability theory. Beurling slow variation has been in use ever since (see BGT §2.11 and [Kor] for background and references). Beurling regular variation was introduced in our recent paper [BinO3]. Here it emerged that the Beurling theory of slow and regular variation *includes* the Karamata theory (despite Beurling slow variation having previously appeared to be a minor topic within the Karamata theory). The role of (CFE) and (GFE) is played in this Beurling context by a functional equation, which we call here the *Beurling functional equation* ((BFE) – see below). This functional equation is a special case of one independently introduced by Aczél and by Gołąb and Schinzel [GolS] in 1959, and (without knowledge of Beurling's work) was called the *Gołąb-Schinzel functional equation* in Aczél and Dhombres [AczD]; see §6.2 for more current information on this literature and associated applications. It has recently been studied by the second author [Ost2] for its relation with uniform convergence in the context of Beurling regular variation; here we unify these two lines of work.

The theme of the present paper is that one begins with the functional inequality, imposes a suitable side-condition (which serves to 'give the inequality the other way') and deduces the corresponding functional equation, which under suitable conditions one is able to solve. The functional equation and functional inequality we have in mind are those mentioned above:

$$F^*(u+v) \leqslant e^{\rho v} F^*(u) + F^*(v) \qquad (\forall u, v \in \mathbb{R}) \qquad (GFI)$$

(BGT (3.2.5), cf. (3.0.11) – for F^* and F see below), and

$$F^*(u+v) = e^{\rho v} K(u) + F^*(v) \qquad (\forall u, v \in \mathbb{R}) \qquad (GFE)$$

(see BGT (3.2.7) – for the relationship here for $\rho \neq 0$ between F^* and its *Goldie kernel* K, indeed in greater generality, see Th. 3 in §2 below; cf. [BinO4, Prop. 1] for the *additive kernel* in the case $\rho = 0$, which reduces (GFI)

to subadditivity on $\mathbb{R}_+$, the context there). (GFI) captures an asymptotic relation in functional form, and so is key to establishing the Characterization Theorem of regular variation (BGT §1.4). Our focus here is on the extent to which the universal quantifiers occurring in the functional inequalities and functional equations under study can be weakened, in the presence of suitable side-conditions. The prototypical side-condition here is the *Heiberg-Seneta condition*

$$\limsup_{u \downarrow 0} F(u) \leqslant 0, \qquad (HS(F))$$

due to Heiberg [Hei] in 1971, and Seneta [Sen] in 1976 (BGT, Th. 3.2.4). This condition, which is best possible here, is what is needed to reduce (GFI) to (GFE).

Two related matters occur here. One is the question of *quantifier weakening* above. This, together with (HS), hinges on the algebraic nature of the set on which one can assert equality. The second, *automatic continuity*, relates to the extent to which a solution of (GFE) is continuous (and hence easily of standard form – see BGT Ch. 3), or (in the most important case $\rho = 0$) an *additive* function becomes *continuous*, and so *linear*. This is the instance of the important subject of *automatic continuity* relevant here. Automatic continuity has a vast literature, particularly concerning homomorphisms of Banach algebras, for which see [Dal1] and [Dal2]. See also Helson [Hel] for Gelfand Theory, Ng and Warner [NgW] and Hoffman-Jørgensen [HJ]. The crux here is the *dichotomy* between additive functions with a hint of regularity, which are then linear, and those without, which are pathological. For background and references on dichotomies of this nature, Hamel pathologies and the like, see [BinO2].

One of our themes here and in [BinO4] is *quantifier weakening*: one weakens a universal quantifier $\forall$ by thinning the set over which it ranges. In what follows we will often have two quantifiers in play, and will replace "$\forall u \in \mathbb{A}$" by "$(u \in \mathbb{A})$", etc. This convention, convenient here, is borrowed from mathematical logic.

One theme that this paper and [BinO4] have in common is the great debt that the subject of regular variation, as it has developed since [BinG] and BGT, owes to the Goldie argument. It is a pleasure to emphasize this here. This argument originated in a study of *Frullani integrals*, important in many areas of analysis and probability ([BinG I, II.6]; cf. BGT §1.6.4, [BinO4, §1]).

2 Generalized Goldie equation

We begin by generalizing (GFE) by replacing the exponential function on the right by a more general function g, the *auxiliary function.* We further generalize by *weakening the quantifiers*, allowing them to range over a set $\mathbb{A}$ smaller than $\mathbb{R}$. It is appropriate to take $\mathbb{A}$ as a dense (additive) subgroup. The functional equation in the result below, written there ($G_{\mathbb{A}}$), may be thought of as the second form of the Goldie functional equation above. As we see in Theorem 1 below, the two coincide in the principal case of interest – compare the insightful Footnote 3 of [BojK]. The notation H_ρ below is from BGT §3.1.7 and 3.2.1 implying $H_0(t) \equiv t$. The identity $uv - u - v + 1 = (1-u)(1-v)$ gives that $(1-e^{-\rho x})/\rho$ is *subadditive* on $\mathbb{R}_+ := (0, \infty)$ for $\rho \geqslant 0$, and *superadditive* on $\mathbb{R}_+$ for $\rho \leqslant 0$.

Theorem 1. *For g with $g(0) = 1$, if $K \neq 0$ satisfies ($G_{\mathbb{A}}$) below with $\mathbb{A}$ a dense subgroup:*

$$K(u + v) = g(v)K(u) + K(v), \qquad (u, v \in \mathbb{A}) \qquad (G_{\mathbb{A}})$$

– then

$$\mathbb{A}_g := \{u \in \mathbb{A} : \ g(u) = 1\}$$

is an additive subgroup on which K is additive, and for some constant κ

$$K(t) \equiv \kappa(g(t) - 1) \text{ for } t \in \{0\} \cup \mathbb{A}\backslash\mathbb{A}_g. \qquad (*)$$

For $\mathbb{A} = \mathbb{R}$ and g locally bounded at 0 with $g \neq 1$ except at 0 : $g(x) \equiv e^{-\rho x}$ for some constant $\rho \neq 0$, and so $K(t) \equiv \kappa\rho H_\rho(t)$, where

$$H_\rho(t) := (1 - e^{-\rho t})/\rho.$$

Proof. Recall that

$$\mathcal{N}_K := \{x \in \mathbb{A} : K(x + a) = K(x) + K(a) \ (\forall a \in \mathbb{A})\}$$

is the *Cauchy nucleus* of K – see [Kuc, §18.5], and is either empty or a subgroup (for a proof see [Kuc, Lemma 18.5.1], or the related [AczD, Ch. 6, proof of Th. 1). If $x \in \mathcal{N}_K$, choosing $a \in \mathbb{A}$ with $K(a) \neq 0$ yields $g(x) = 1$ from

$$K(a + x) = K(a) + K(x) = g(x)K(a) + K(x).$$

Conversely, for $v \in \mathbb{A}_g$ and any $v \in \mathbb{A}$

$$K(u+v) = K(u) + K(v)$$

so $v \in \mathcal{N}_K$: $\mathbb{A}_g = \mathcal{N}_K$. By assumption $0 \in \mathbb{A}_g$, so in particular K is additive on $\mathbb{A}_g$, and $K(0) = 0$.

Continue now as in BGT Lemma 3.2.1: for $u, v \in \mathbb{A} \backslash \mathbb{A}_g$ distinct:

$$g(v)K(u) + K(v) = K(u+v) = K(v+u) = g(u)K(v) + K(u).$$

So

$$K(u)[g(v)-1] = K(v)[g(u)-1] : \qquad \frac{K(u)}{g(u)-1} = \frac{K(v)}{g(v)-1} = \kappa,$$

say; so, for some constant κ,

$$K(u) = \kappa[g(u)-1]$$

on $\mathbb{A} \backslash \mathbb{A}_g$, proving $(*)$ in this case. Although we assumed $u \neq 0$, we still have $0 = K(0) = \kappa[g(0)-1]$, completing the proof of $(*)$.

Substitution (for $u, v \in \mathbb{A} \backslash \mathbb{A}_g$ provided $u + v \in \mathbb{A} \backslash \mathbb{A}_g$) yields first

$$\kappa[g(u+v)-1] = \kappa g(v)[g(u)-1] + \kappa[g(v)-1],$$

and then, for $\kappa \neq 0$, the Cauchy exponential equation

$$g(u+v) = g(v)g(u), \qquad (CEE)$$

if $\mathbb{A}_g = \{0\}$; so if $\mathbb{A} = \mathbb{R}$, local boundedness yields $g(x) \equiv e^{-\rho x}$ for some ρ (see [AczD, Ch. 3], or [Kuc, §13.1]). If $\kappa = 0$ above, then $K(x) \equiv 0$. $\square$

Remarks. 1. In Theorem 1 above the additive reals act on the domain of the unknown function K. Generalizations are possible to other group actions and will rely on the auxiliary function g being a group homomorphism as in (CEE) above. There is more to be said here; we hope to this elsewhere.
2. Recall that f satisfies the *Mikusiński equation* if

$$f(x+y) = f(x) + f(y) \qquad \text{if } f(x+y) \neq 0; \qquad (Mik)$$

such a function is necessarily additive, for which see [AczD, Ch. 6 Th. 1]. The argument above identifies g from (CEE) when $u, v \in \mathbb{A} \backslash \mathbb{A}_g$, provided

$u+v \in \mathbb{A}\backslash\mathbb{A}_g$; so for $g>0$, the condition $g(u+v) \neq 1$ is equivalent to $\log g \neq 0$, which means that $\log g$ satisfies (Mik) and so $g(x) = \exp f(x)$ for some additive (possibly pathological) function f.
3. Above, for g Baire/measurable, by the Steinhaus subgroup theorem (see e.g. [BinO2, Th. S] for its general combinatorial form), $\mathbb{A}_g = \mathbb{R}$ iff $\mathbb{A}_g$ is non-negligible, in which case K is additive. The additive case is studied in [BinO4] and here we have passed to $\mathbb{A}_g = \{0\}$ as a convenient context. More, however, is true. As an alternative to the last remark, for $\mathbb{A}_g$ negligible: by the Fubini/Kuratowski-Ulam Theorem (for which see [Oxt, Ch. 14-15]), the equation (CEE) above holds for quasi almost all $(u,v) \in \mathbb{R}^2$; consequently, by a theorem of Ger (see [Ger], or [Kuc, Th.18.71]), there is a homomorphism on $\mathbb{A}$ 'essentially extending' $\log g$ to $\mathbb{A}$. From here, again for g Baire/measurable, $g(x) = e^{-\rho x}$ for some ρ.

In Theorem 2 below, there is no quantifier weakening to $\mathbb{A}$ and so we need $(G_{\mathbb{R}})$ in place of $(G_{\mathbb{A}})$. It will be convenient in what follows to write 'positive' for functions to mean 'positive on $\mathbb{R}_+$', unless otherwise stated.

Theorem 2. *If both K and g in $(G_{\mathbb{R}})$ are positive with $g \neq 1$ except at 0, then either $K \equiv 0$, or both are continuous, and $g(x) \equiv e^{-\rho x}$ for some $\rho \neq 0$.*

Proof. Writing $w = u+v$, one has

$$K(w) - K(v) = g(v)K(w-v),$$

so K is strictly increasing and so continuous at some point $y>0$ say. But for any h

$$K(y+h) - K(y) = g(y)K(h),$$

and so, since $g(y) > 0$, K is continuous at 0, as $K(0) = 0$. Hence K is continuous at any point $t > 0$, since $g(t) > 0$ and

$$K(t+h) - K(t) = g(t)K(h).$$

Take any $u > 0$. Fix $w > u$, so that $K(w-u) > 0$. Then, since

$$g(u) = [K(w) - K(u)]/K(w-u),$$

and the right-hand side is continuous in u for $u > 0$, the function g is continuous for $u > 0$. Finally, as in Th. 1, $K(x) = \kappa[g(x) - 1]$ for all x, for some constant κ; if $\kappa \neq 0$, then g satisfies (CEE) and is continuous on $\mathbb{R}_+$, so again the function g is $e^{-\rho x}$ and K is continuous. $\square$

Remark. For $g \equiv 1$ in $(G_{\mathbb{R}})$, the proof of Theorem 2 shows that a positive additive function is continuous – a weak form of Darboux's Theorem on the continuity of bounded additive functions.

In $(G_{\mathbb{R}})$ above for $x, \rho \geqslant 0$ one has $g(x) = e^{-\rho x} \leqslant 1$ on $\mathbb{R}_+$; generally, if $g(x) \leqslant 1$ on $\mathbb{R}_+$ and K positive satisfies $(G_{\mathbb{R}})$, then for $u, v \geqslant 0$

$$K(u+v) \leqslant K(u) + K(v),$$

and so K is subadditive on $\mathbb{R}_+$.

We now prove a converse – our main result. Here, in the context of subadditivity, the important role of the *Heiberg-Seneta condition*, discussed in §1, is performed by a weaker side-condition: *right-continuity at* 0, a consequence, established in [BinG] – see also BGT §3.2.1 and [BinO3]. A further quantifier weakening occurs in (ii) below.

Theorem 3 (Generalized Goldie Theorem). *If for $\mathbb{A}$ a dense subgroup,*
(i) $F^* : \mathbb{R} \to \mathbb{R}$ *is positive and subadditive with* $F^*(0+) = 0$;
(ii) F^* *satisfies the weakened Goldie equation*

$$F^*(u+v) = g(v)K(u) + F^*(v) \qquad (u \in \mathbb{A})(v \in \mathbb{R}_+)$$

for some non-zero K satisfying $(G_{\mathbb{A}})$ with g continuous and $\mathbb{A}_g = \{0\}$;
(iii) F^* *extends K on* $\mathbb{A}$:

$$F^*(x) = K(x) \text{ for } x \in \mathbb{A},$$

so that in particular F^* satisfies $(G_{\mathbb{A}})$, and indeed

$$F^*(u+v) = g(v)F^*(u) + F^*(v) \text{ for } u \in \mathbb{A}, v \in \mathbb{R}_+;$$

– then for some $c > 0, \rho \geqslant 0$

$$g(x) \equiv e^{-\rho x} \text{ and } F^*(x) \equiv cH_\rho(x) = c(1 - e^{-\rho x})/\rho.$$

Proof. Put

$$\gamma(x) = \int_0^x g(t)dt : \qquad \gamma'(x) = g(x).$$

By continuity of g and Th. 1, $K(u+) = K(u)$ for all $u \in \mathbb{A}$, and so $K(0+) = 0$. Furthermore, note that F^* is right-continuous on $\mathbb{A}$ (and $F^*(u+) = K(u)$ on $\mathbb{A}$), and on $\mathbb{R}_+$

$$\limsup_{v \downarrow 0} F^*(u+v) \leqslant F^*(u) + F^*(0+) = F^*(u).$$

Now proceed as in the Goldie proof – see e.g. BGT §3.2.1. (This uses the sequence $s_n = n\delta$, rather than the Beck sequence of §3 below which is not appropriate here, but see below in Theorem 7 for a Beck-sequence adaptation of the current argument.) For any u, u_0 with $u_0 \in \mathbb{A}$ and $u_0 > 0$, define $i = i(\delta) \in \mathbb{Z}$ for $\delta > 0$ so that $(i-1))\,\delta \leqslant u < i\delta$, and likewise for u_0 define $i_0(\delta)$. Also put

$$c_0 := K(u_0)/[g(u_0) - 1].$$

As $m\delta \in \mathbb{A}$,

$$F^*(m\delta) - F^*((m-1)\delta) = g((m-1)\delta)K(\delta),$$

so that on summing

$$F^*(i(\delta)\delta) = K(\delta) \sum_{m=1}^{i} g((m-1)\delta), \tag{**}$$

as $F^*(0) = 0$. Note that as $\delta \to 0$,

$$\delta \sum_{m=1}^{i} g((m-1)\delta) \to \int_0^u g(x)dx \tag{RI}$$

(for 'Riemann Integral'). Assuming we may take limits as $\delta \to 0$ through positive $\delta \in \mathbb{A}$ with $K(\delta) \neq 0$, we then have

$$\frac{F^*(i(\delta)\delta)}{F^*(i_0(\delta)\delta)} = \frac{K(\delta)}{K(\delta)} \frac{\sum_{m=1}^{i} g((m-1)\delta)}{\sum_{m=1}^{i_0} g((m-1)\delta)} = \frac{\delta \sum_{m=1}^{i} g((m-1)\delta)}{\delta \sum_{m=1}^{i_0} g((m-1)\delta)} \to \frac{\gamma(u)}{\gamma(u_0)}.$$

So by right-continuity at u_0,

$$\lim F^*(i_0(\delta)\delta) = F^*(u_0) = K(u_0) = c_0[g(u_0) - 1].$$

So

$$F^*(i(\delta)\delta) \to \gamma(u) \cdot F^*(u_0)/\gamma(u_0).$$

As before, as $u_0 \in \mathbb{A}$,

$$\begin{aligned} F^*(u) &\geqslant \limsup F^*(i(\delta)\delta) = \gamma(u) \cdot F^*(u_0)/\gamma(u_0) \\ &= \gamma(u)K(u_0)/\gamma(u_0) = \gamma(u)c_0[g(u_0) - 1]/\gamma(u_0). \end{aligned}$$

Put

$$c_1 = c_0[g(u_0) - 1]/\gamma(u_0).$$

Now specialize to $u \in \mathbb{A}$, on which, by above, F^* is right-continuous. Letting $i(\delta)\delta \in \mathbb{A}$ decrease to u, the inequality above becomes an equation:

$$K(u) = F^*(u) = \gamma(u)c_0[g(u_0) - 1]/\gamma(u_0) = c_1\gamma(u).$$

This result remains valid with $c_1 = 0$ if $K(\delta) = 0$ for $\delta \in \mathbb{A} \cap I$ for some interval $I = (0, \varepsilon)$, as then $F^*(u) = 0$ by right-continuity, because $F^*(i(\delta)\delta) = 0$ for $\delta \in \mathbb{A} \cap I$, by (**).

Now for *arbitrary* u, taking $v \uparrow u$ with $v \in \mathbb{A}$, we have (as $u - v > 0$) that

$$\begin{aligned} F^*(u) &= F^*(u - v + v) = K(v)g(u - v) + F^*(u - v) \qquad \text{(by (ii), as } v \in \mathbb{A}) \\ &= c_1\gamma(v)g(u - v) + F^*(u - v) \to c_1\gamma(u), \end{aligned}$$

by continuity of γ. Thus for *all* u,

$$F^*(u) = c_1\gamma(u).$$

Thus by Theorem 1, for some κ

$$c_1\gamma(u) = F^*(u) = K(u) = \kappa[g(u) - 1]$$

on $\mathbb{A}$. So, by density and continuity on $\mathbb{R}_+$,

$$\kappa[g(u) - 1] = c_1\gamma(u).$$

So g is indeed differentiable; differentiation now yields

$$c_1 g(u) = \kappa g'(u).$$

If $\kappa = 0$, then $K(u) \equiv 0$, contrary to assumptions. So

$$g'(u) = (c_1/\kappa)g(u),$$

and so with $\rho := -c_1/\kappa$

$$g(u) = e^{-\rho u} \text{ and } \gamma(u) = H_\rho(u): \qquad F^*(u) = c_1\gamma(u) = c_1[1 - e^{-\rho u}]/\rho.$$

Finally, as $(1-e^{-\rho x})/\rho$ is subadditive iff $\rho \geqslant 0$ (cf. before Theorem 1), $c_1 > 0$. $\square$

Remark. From the passage from the Riemann sum in (RI) to the Riemann integral, we see the origin of the otherwise surprising feature: that we obtain automatic differentiability from continuity in several of the arguments below.

Theorem 4. *If g, K are positive, F^* is subadditive on $\mathbb{R}_+$ with $F^*(0+) = 0$, and*

$$F^*(u + v) = g(v)K(u) + F^*(v) \qquad (u \in \mathbb{A})(v \in \mathbb{R}_+)$$

– then F^* is increasing and continuous on $\mathbb{R}_+$, and so g is continuous on $\mathbb{R}_+$. In particular, the continuity assumed in Theorem 3 above is implied by positivity of both g and K.

Proof. Since

$$F^*(v + u) - F^*(v) = g(v)K(u) \qquad (u \in \mathbb{A})(v \in \mathbb{R}_+),$$

then for $u > 0$ and $u \in \mathbb{A}$

$$F^*(v + u) > F^*(v) \qquad (v \in \mathbb{R}).$$

So letting $u \downarrow 0$ through $\mathbb{A}$,

$$F^*(v) \leqslant \lim \sup_{u\downarrow 0 \text{ in } \mathbb{A}} F^*(v+u) \leqslant \lim\sup_{u\downarrow 0} F^*(v+u) \leqslant F^*(v)+F^*(0+) = F^*(v),$$

and so

$$F^*(v+) = F^*(v),$$

i.e. F^* is right-continuous everywhere on $\mathbb{R}_+$. Now for $u \in \mathbb{A}$ with $0 < u < w$

$$F^*(w - u) < F^*((w - u) + u) = F^*(w).$$

So, for arbitrary $0 < v < w$, and $u \in \mathbb{A}$ with $u > 0$ such that $v < w - u < w$,

$$F^*(v) = F^*(v+) = \lim\inf\{F^*(w - u) : v < w - u < w, u \in \mathbb{A}\} < F^*(w),$$

as $\mathbb{A}$ is dense. So

$$F^*(v) < F^*(w),$$

i.e. F^* is increasing. So it is continuous at some $v > 0$. Since $K(0) = 0$, K is continuous at 0 on $\mathbb{A}$. But

$$F^*(v-u) - F^*(v) = g(v)K(-u) \qquad (u \in \mathbb{A})(v \in \mathbb{R}),$$

so F^* satisfies, for any $v \in \mathbb{R}_+$,

$$\lim_{u \downarrow 0 \ \& \ u \in \mathbb{A}} F^*(v-u) = F^*(v).$$

But F^* is increasing, and $\mathbb{A}$ is dense; so F^* is continuous. Then for any fixed $u > 0$ in $\mathbb{A}$,

$$g(v) = [F^*(v+u) - F^*(v)]/K(u),$$

and so g is continuous at any $v \in \mathbb{R}_+$, since F^* is continuous at v and at $v + u$. $\square$

3 From the Goldie to the Beurling Equation

In (GFE), take K and F^* (which will reduce to the same – see Th. 5 below) as K. We generalize the $e^{-\rho \cdot}$ to g, which will serve as an *auxiliary function* (which will reduce to $e^{-\rho \cdot}$ in the case of interest). We now have the Goldie equation in the form

$$K(v+u) - K(v) = g(v)K(u).$$

For reasons that will emerge (see inter alia §5), an important generalization arises if on the left the additive action of v on u is made dependent on g:

$$K(v + ug(v)) - K(v) = g(v)K(u), \tag{1}$$

so that while g appears twice, K still appears here three times. This form is closely related to a situation with all function symbols identical, φ say (which we will take non-negative):

$$\varphi(v + u\varphi(v)) = \varphi(u)\varphi(v), \qquad (\forall u, v \in \mathbb{R}_+). \tag{BFE}$$

Indeed, from here, writing g for φ and with $K(t) \equiv g(t) - 1$ (i.e. as in (*) with $\kappa = 1$), we recover (1).

This (BFE) is our *Beurling functional equation,* a special case of the Gołąb-Schinzel equation (see §1) in view of the non-negativity and of the domain being $\mathbb{R}_+$ rather than $\mathbb{R}$ (both considerations arising from the context of Beurling regular variation). Solutions of the 'conditional' Gołąb-Schinzel equation (i.e. with domain restricted to $\mathbb{R}_+$, but without the non-negativity restriction) were considered and characterized in [BrzM] and shown to be extendible uniquely to solutions with domain $\mathbb{R}$. Note that for any extension to $\mathbb{R}_+ \cup \{0\}$, if $\varphi(0) = 0$, then (BFE) implies $\varphi = 0$; we will therefore usually set $\varphi(0) = 1$, the alternative dictated by the equation $\varphi(0) = \varphi(0)^2$. Solutions $\varphi > 0$ are relevant to the Beurling theory of regular variation – see [Ost2] for an analysis; their study is much simplified by the following easy result, inspired by a close reading of [Brz1, Prop. 2].

Theorem 5. *If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (BFE), then $\varphi(x) \geqslant 1$ for all $x > 0$.*

Proof. Suppose that $\varphi(u) < 1$ for some $u > 0$; then $v := u/(1 - \varphi(u)) > 0$ and so, since $v = u + v\varphi(u)$,

$$0 < \varphi(v) = \varphi\left(u + v\varphi(u)\right) = \varphi(u)\varphi(v).$$

So cancelling $\varphi(v)$, one has $\varphi(u) = 1$, a contradiction. □

The theorem above motivates the introduction of an important tool in the study of positive solutions φ: the *Beck sequence* $t_m = t_m(u)$, defined for any $u > 0$ recursively by

$$t_{m+1} = t_m + u\varphi(t_m) \text{ with } t_0 = 0,$$

so that

$$\varphi(t_{m+1}) = \varphi(u)\varphi(t_m).$$

By Theorem 5, the sequence t_m is divergent, since either $\varphi(u) = 1$ and $t_m = mu$, or else

$$t_m = u\frac{\varphi(u)^m - 1}{\varphi(u) - 1} = (\varphi(u)^m - 1) \Big/ \frac{\varphi(u) - 1}{u}\ , \tag{2}$$

e.g. by Lemma 4 of [Ost2] (cf. a lemma of Bloom: BGT Lemma 2.11.2). In either case, for $u, t > 0$ a unique integer $m = m_t(u)$ exists satisfying

$$t_m \leqslant t < t_{m+1}.$$

This tool will enable us to prove in Theorem 7 below that a positive solution of (BFE) takes the form $\varphi(t) = 1 + ct$ for some $c \geqslant 0$. Theorem 6 and its Corollary below lay the foundations.

Theorem 6. *If a function $\varphi \geqslant 0$ satisfies the equation (BFE) on $\mathbb{R}_+$ with $\varphi(t) > 1$ for $t \in I = (0, \delta)$, for some $\delta > 0$, then φ is continuous and (strictly) increasing, and $\varphi > 1$.*

Proof. Take $K(t) = \varphi(t) - 1$; then $K > 0$ on I. Writing $x = u$ and $y = v\varphi(u)$,

$$\varphi(x + y) - \varphi(x) = K(y/\varphi(x))\varphi(x).$$

Fix $x \in I$; then $\varphi(x) > 1$, and so $y/\varphi(x) \in I$ for $y \in I$, so that $K(y/\varphi(x)) > 0$. As in Theorem 2, $\varphi(x + y) > \varphi(y)$ for $x, y \in I$, and φ is increasing on a subinterval of I. So φ is continuous at some point $u \in I$, $\varphi(u) > 0$ and

$$\varphi(u) = \lim_{v \downarrow 0} \varphi(u + v\varphi(u)) = \varphi(u) \lim_{v \downarrow 0} \varphi(v) : \qquad \varphi(0+) = \lim_{v \downarrow 0} \varphi(v) = 1.$$

So for $x > 0$ with $\varphi(x) > 0$,

$$\lim_{v \downarrow 0} \varphi(x + v\varphi(x)) = \varphi(x) \lim_{v \downarrow 0} \varphi(v) = \varphi(x),$$

and so φ is right-continuous at x.

Let $J \supseteq I$ be a maximal interval $(0, \eta)$ on which φ is increasing, and suppose that η is finite. Then $\varphi(\eta) = 0$: otherwise, $\varphi(\eta) > 0$ and as above φ is right-continuous at η; since $\varphi > 1$ in I, $\varphi(t) > 1$ near η; then φ is increasing to the right of η, a contradiction. Now choose $t < \eta < t + \delta$; then $v = (\eta - t)/\varphi(t) < x - t$, and

$$\varphi(\eta) = \varphi(t + v\varphi(t)) = \varphi(t)\varphi(v) > 0,$$

as $t \in J$, contradicting $\varphi(\eta) = 0$. This shows that $J = \mathbb{R}_+$, and so φ is right-continuous and increasing.

Finally, we check that φ is left-continuous at any $x > 0$. Let $z_n \downarrow 0$ with $x - z_n > 0$; then, as above, $\varphi(z_n) \to 1$. As φ is increasing, $u_n := \varphi(x - z_n) - 1 \leqslant \varphi(x) - 1$ is positive and bounded, so

$$\begin{aligned} \varphi(x - z_n) &= \varphi((x - z_n) + z_n\varphi(x - z_n))/\varphi(z_n) \\ &= \varphi(z + z_n u_n)/\varphi(z_n) \to \varphi(x), \end{aligned}$$

by right-continuity at x. □

Corollary. *If* $\varphi > 0$, *then* φ *is continuous, and either* $\varphi > 1$, *or the value 1 is repeated densely and so* $\varphi \equiv 1$.

Proof. By Th. 5 $\varphi \geqslant 1$, so φ is (weakly) increasing and so continuous. Suppose that $\varphi > 1$ is false; then, by Theorem 6, there is no interval $(0, \delta)$ with $\delta > 0$ on which $\varphi > 1$, and so there are arbitrarily small $u > 0$ with $\varphi(u) = 1$. Fix $t > 0$. For any u with $\varphi(u) = 1$, choose $n = n_t(u)$ with $t_n := nu \leqslant t \leqslant (n+1)u$, as above. Then $\varphi(t_n) = 1$ and $0 \leqslant t - t_n < u$. So the value 1 is taken on a dense set of points, and so by continuity $\varphi(t) \equiv 1$. □

We now adapt Goldie's argument above to give an easy proof of the following. Theorem 7 below can be derived from [Brz1, Cor 3] or [Brz2, Th1]. There algebraic considerations are key; an analytical proof was provided in [Ost2], but by a different and more complicated route. We include the proof below for completeness, as it is analogous to the Goldie Theorem above and so thematic here. We use a little less than Theorem 6 provides.

Theorem 7. *If* $\varphi(t) > 1$ *holds for all* t *in some interval* $(0, \delta)$ *with* $\delta > 0$, *and satisfies* (BFE) *on* $\mathbb{R}_+$, *then* φ *is differentiable, and takes the form*

$$\varphi(t) = 1 + ct.$$

Proof. Fix $x_0 > 0$ with $\varphi(x_0) \neq 1$. Put

$$K(t) := \varphi(t) - 1.$$

By Th. 6 K is continuous, so $K(t) \neq 0$ for t sufficiently close to x_0; we may assume also that $\varphi(u) \neq 1$ for all small enough $u > 0$, and so $K(u) \neq 0$ for sufficently small u.

Let x be arbitrary; in the analysis below x and x_0 play similar roles, so it will be convenient to also write x_1 for x.

For $j = 0, 1$ and any $u > 0$, referring to the Beck sequence $t_m = t_m(u)$ as above, select $i_j = i_j(u) := m_{x_j}(u)$ so that

$$t_{i_j} \leqslant x_j < t_{i_j+1} \qquad (j = 0, 1);$$

then

$$\varphi(t_{m+1}) - \varphi(t_m) = \varphi(u)\varphi(t_m) - \varphi(t_m) = K(u)\varphi(t_m).$$

Summing,

$$\varphi(v_m) - \varphi(v_0) = K(u) \sum_{n=0}^{m-1} \varphi(t_n).$$

As noted, for all small enough u, $K(v(i_0))$ non-zero (this use compactness of $[0, x_0]$). Cancelling $K(u)$ below (as also $K(u)$ is non-zero), and introducing u in its place (to get the telescoping sums),

$$\frac{K(t_{i_1})}{K(t_{i_0})} = \frac{K(u) \sum_{n=0}^{i_1-1} \varphi(t_n)}{K(u) \sum_{n=0}^{m-1} \varphi(t_n)} = \frac{\sum_{n=0}^{i_1-1} u\varphi(t_n)}{\sum_{n=0}^{m-1} u\varphi(t_n)} = \frac{\sum_{n=0}^{i_1-1} t_{n+1} - t_n}{\sum_{n=0}^{i_0-1} t_{n+1} - t_n} = \frac{t_{i_1}}{t_{i_0}},$$

as

$$t_{n+1} - t_n = u\varphi(t_n).$$

Passing to the limit as $u \to 0$, by continuity

$$K(x_1)/K(x_0) = x_1/x_0.$$

Setting $c_0 := K(x_0)/x_0$,

$$\varphi(x) - 1 = K(x) = c_0 x : \ \varphi(x) = 1 + c_0 x. \ \square$$

Remark. The argument above could have been presented explicitly in terms of integrals. Relevant here is the differentiability referred to above.

4 On the Beurling functional equation

This section is suggested by the recent work of the second author [Ost2]. We include it here for two reasons. First, it is thematically close to other results in this paper. Secondly, it provides a simpler proof of results concerning Beurling's equation than through specialization of results by Brzdęk and by Brzdęk and Mureńko in [Brz1] and [BrzM]. Theorem B at the end of this section is taken from these papers; we include it here for completeness, and as our proof is more direct and shorter.

In Th. 8 below, with context as in Ths. 5 and 6, we use λ rather than φ for ease of comparison with [Ost2]. For $\lambda : [0, \infty) \to \mathbb{R}$, denote its level set above unity by:

$$L_+(\lambda) := \{t \in \mathbb{R}_+ : \lambda(t) > 1\}.$$

Theorem 8. *If the continuous solution λ of (BFE) with $\lambda(0) = 1$ has a nonempty level set $L_+(\lambda)$ containing an interval $(0, \delta)$ for some $\delta > 0$, then λ is differentiable and for some $\rho > 0$*

$$\lambda(t) \equiv 1 + \rho t.$$

Proof. We recall (from the proof of Theorem 5 above) the Beck sequence $t_n(u)$, and that $\lambda(t_n(u)) = \lambda(u)^n$; from here, for $u \in L_+$, by summing,

$$t_n(u) = u\frac{\lambda(u)^n - 1}{\lambda(u) - 1} = (\lambda(u)^n - 1) \Big/ \frac{\lambda(u) - 1}{u} ,$$

(and with $t_n(u) = nu$ if $\lambda(u) = 1$), and from the recurrence,

$$\Delta_m(u) := t_{m+1}(u) - t_m(u) = u\lambda(u)^m.$$

For $T \in L_+$ and $u > 0$, write $m = m(u) = m_T(u)$ for the jump index for which

$$t_m(u) \leqslant T < t_{m+1}(u).$$

By (2) and continuity at 0 of λ,

$$\Delta_{m(u)}(u) = u\lambda(u)^{m(u)} \leqslant T(\lambda(u) - 1) + u \to 0 \text{ as } u \to 0, \tag{3}$$

for $u \in L_+$ *uniformly* in $T > 0$ on compacts. Likewise for $u \notin L_+$, as then $\Delta_{m(u)}(u) = u$.

Consider any null sequence $u_n \to 0$ with $u_n > 0$. We will show that $\{(\lambda(u_n) - 1)/u_n\}$ is convergent, by showing that down every subsequence $\{(\lambda(u_n) - 1)/u_n\}_{n \in \mathbb{M}}$ there is a convergent sub-subsequence with limit independent of $\mathbb{M}$.

W.l.o.g. we take $0 < u_n \in L_+$ for all n (so $u_n < \delta$). Now consider an arbitrary $T \in L_+$. Passing to a subsequence (dependent on T) of $\{(\lambda(u_n) - 1)/u_n\}_{n \in \mathbb{M}}$ if necessary, we may suppose, for $k(n) := m_T(u_n)$, that $\Delta_{k(n)}(u_n) \to 0$; then along $\mathbb{M}$

$$|T - t_{m(u_n)}(u_n)| \leqslant \Delta_{m(u_n)}(u_n), \text{ and so } t_{k(n)}(u_n) = t_{m(u_n)}(u_n) \to T.$$

Again by (2) and continuity at T of λ, putting $\rho := (\lambda(T) - 1)/T > 0$,

$$\frac{\lambda(u_n) - 1}{u_n} = \frac{\lambda(u_n)^{m(u_n)} - 1}{t_{m(u_n)}(u_n)} = \frac{\lambda(t_{m(n)}(u_n)) - 1}{t_{m(u_n)}(u_n)} \to \frac{\lambda(T) - 1}{T} = \rho,$$

along $\mathbb{M}$ to a limit ρ dependent only on T. So $\{(\lambda(u_n) - 1)/u_n\}$ is itself convergent to ρ. But this holds for any null sequence $\{u_n\}$ in $\mathbb{R}_+$, so the function λ is differentiable at 0, and so is right-differentiable everywhere in L_+ (see [Ost2, Lemma 3]). It is also left-differentiable at any $x > 0$, as follows. For y with $0 < y < x$, put $t := (x - y)/\lambda(y) > 0$. Then $x = y + t\lambda(y)$, so

$$\frac{\lambda(x) - \lambda(y)}{x - y} = \frac{\lambda(y + t\lambda(y)) - \lambda(y)}{x - y} = \frac{[\lambda(t) - 1]\lambda(y)}{x - y} = \frac{\lambda(t) - 1}{t}.$$

But $t \downarrow 0$ as $y \uparrow x$ (by continuity of λ at x), and $(\lambda(t) - 1)/t \to \lambda'(0)$. So λ is left-differentiable at x and so differentiable; from here $\lambda'(x) = \lambda'(0)$.

Integration then yields $\lambda(x)$; also, since T above was arbitrary, for any $T \in L_+$

$$\rho = \lim_{n\in\mathbb{M}}\{(\lambda(u_n) - 1)/u_n\} = \lambda'(0) = (\lambda(T) - 1)/T : \qquad \lambda(x) = 1 + \rho x \ (x \in \mathbb{R}_+). \ \square$$

In Theorem BM below we use f rather than φ for ease of comparison with [BrzM].

Theorem BM ([BrzM, Lemma 7])**.** *For $f > 0$ a solution of (BFE), if $f \neq 1$ at all points, then $f(x) = 1 + cx$ for some $c > 0$.*

Proof. By symmetry, for any x, y

$$f(x + yf(x)) = f(x)f(y) = f(y + xf(y)).$$

Fix x and y and put $u = x + yf(x)$ and $v = y + xf(y)$. If these are unequal, w.l.o.g. suppose that $v > u$. Then $(v - u)/f(u) > 0$, so

$$0 < f(u) = f(v) = f(u + f(u)(u - v)/f(u)) = f(u)f((u - v)/f(u)).$$

So $f((u - v)/f(u)) = 1$, a contradiction. So $u = v$: that is, for all $x, y > 0$

$$x + yf(x) = y + xf(y);$$

equivalently, for all $x, y > 0$

$$x/(1 - f(x)) = y/(1 - f(y)) = c,$$

say. Then $f(x) = 1 + cx$ for all $x > 0$. So $c > 0$. $\square$

Below we suppose that $f(a) = 1$, for some fixed $a > 0$. Note that $t_n := na$ is a *Beck* sequence with step size a; so $f(na) = 1$, since $f(t_n) = f(t_1)^n$.

For $f > 0$ a solution of (BFE), we denote here the range of f by $R_f := \{w : (\exists x > 0) w = f(x)\}$. If $f \equiv 1$, then $R_f = \{1\}$.

Lemma B ([Brz1, Cor.1], cf. [BrzM, Lemmas 1,2])**.** *If the value* 1 *is achieved by a solution* $f > 0$ *of* (BFE), *then*
(i) *the range set* R_f *is a multiplicative subgroup;*
(ii) $f(x + a) = f(x)$ for all $x > 0$;
(iii) $f(wa) = 1$ for $w \in R_f$.

Proof. For (i), (BFE) itself implies that R_f is a semigroup. We only need to find the inverse of $w := f(x)$ with $x > 0$. Choose $n \in \mathbb{N}$ with $na > x$. Put $y = (na - x)/f(x)$; then

$$f(x)f(y) = f(x + yf(x)) = f(na) = 1.$$

For (ii), note that $f(x)f(a) = f(a + xf(a)) = f(x + a)$. For (iii), this time write $w = 1/f(x)$; then

$$f(x) = f(x + a) = f(x + f(x)a/f(x)) = f(x)f(aw),$$

and cancelling $f(x) > 0$ gives $f(aw) = 1$. $\square$

Theorem B ([Brz1, Th. 3])**.** *If* $1 \in R_f$, *then* $f \equiv 1$.

Proof. Suppose otherwise; then, by Theorem 5, $f(u) > 1$, for some $u > 0$. Choose $n \in \mathbb{N}$ with $na > u/(f(u) - 1) > 0$, and put

$$v := na + u/(1 - f(u)) > 0 : \; v + naf(u) = na + u + vf(u).$$

So, since $f(u) \in R_f$, applying Lemma B (first (ii), then (i))

$$\begin{aligned} 0 \;&<\; f(v) = f(v + f(u)na) = f(u + vf(u) + na) \\ &=\; f(u + vf(u)) = f(u)f(v), \end{aligned}$$

yielding the contradiction $f(u) = 1$. Hence $f(x) = 1$ for all x. $\square$

5 Extensions of the Goldie and Beurling equations

Below we consider two generalizations of the Beurling equation inspired by Goldie's equation, relevant to Beurling regular variation, for which see [BinO3]. The first uses three functions:

$$K(v+uk(v))-K(v)=g(u)k(v) \qquad (u,v\in\mathbb{R}_+) \qquad (GBE)$$

Here the choice $k=K=\varphi$ with $g=\varphi(u)-1$ recovers the Beurling equation. One can also form a Pexider-like generalization (for which see [Kuc,13.3], or [AczD, 4.3]) for the right-hand side above, by replacing the occurrence of k there with an additional function h :

$$K(v+uk(v))-K(v)=g(u)h(v). \qquad (u,v\in\mathbb{R}_+) \qquad (GBE\text{-}P)$$

Here $h=K$ and $k=1$ yields Goldie's equation; $h=k=K, g=1-k$ yields the Beurling equation.

Theorem 9. *Consider the functional equation $(GBE\text{-}P)$ with $g(u)\neq 0$ for $u>0$ near 0, h,k continuous on $\mathbb{R}_+\cup\{0\}$ and positive, and with $k(0)>0$. With*

$$H(x):=\int_0^x h(t)\frac{dt}{k(t)},\ \text{for } x\geqslant 0,$$

any solution K is differentiable and takes the form $K(x)=cH(x)$, for c a constant; furthermore, g is continuous with $g(0+)=0$ and $h(0)=0$.

Proof. Since

$$K(v+w)-K(v)=g(w/k(v))h(v),$$

we deduce for $u,v\in\mathbb{R}_+$ that

$$K(v+)-K(v)=g(0+)h(v), \qquad K(u-)-K(u)=g(0+)h(u),$$

and that for any $v>0$ and all small enough $w>0$

$$K(v+w)>K(v).$$

So K is locally increasing on $\mathbb{R}_+$, and so is increasing and so continuous on a dense set $D\subseteq\mathbb{R}_+$. For $v\in D$,

$$g(0+)h(v)=K(v+)-K(v)=0,$$

and since $h > 0$, $g(0+) = 0$. So K is continuous on $\mathbb{R}_+$.

Now consider for $u > 0$ the Beck sequence

$$t_{n+1}(u) = t_n(u) + uk(t_n(u)),\ t_0 = 0,$$

which is increasing as $k > 0$. For any $t, u > 0$ we claim there is $m = m_t(u)$ with

$$m_t(u) \leqslant t < m_t(u) + 1.$$

For otherwise, with t, u fixed as above, the sequence $t_n(u)$ is bounded by t and, putting $\tau := \sup t_n(u) \leqslant t$,

$$k(t_n(u) = \frac{1}{u}[t_{n+1}(u) - t_n(u)] \to 0,$$

contradicting lower boundedness of k near τ. Next observe that, since k is bounded on $[0, t]$, by M_t say,

$$t_{m+1}(u) - t_m(u) = uk(t_k(u) \leqslant uM_t \to 0.$$

Now fix x_0,$x_1 > 0$. Select $i_0 = i_0(u)$ and $i_1 = i_1(u)$ so that

$$t_{i_j} \leqslant x_j < t_{i_j+1}.$$

Then

$$K(t_{m+1}) - K(t_m) = g(u)h(t_m).$$

Summing, and setting $p(t) := h(t)/k(t)$,

$$K(t_m) - K(t_0) = g(u) \sum_{n=0}^{m-1} h(t_n) = \frac{g(u)}{u} \sum_{n=0}^{m-1} uk(t_n)p(t_n).$$

For all small enough u we have $g(u)$ non-zero, so

$$\frac{K(t_{i_1})}{K(t_{i_0})} = \frac{g(u)\sum_{n=0}^{i_1-1} h(t_n)}{g(u)\sum_{n=0}^{m-1} h(t_n)} = \frac{\sum_{n=0}^{i_1-1} uk(t_n)p(t_n)}{\sum_{n=0}^{m-1} uk(t_n)p(t_n)} \to \frac{\int_0^{x_i} p(t)dt}{\int_0^{x_0} p(t)dt}.$$

Passing to the limit as $u \to 0$, by continuity of K,

$$K(x_1)/K(x_0) = H(x_1)/H(x_0).$$

Setting $c_0 := K(x_0)/H(x_0)$, we have

$$K(x) = c_0 H(x),$$

for $x \geqslant 0$, which is differentiable.

Substitution in (GBE-P) gives

$$g(u) = \frac{1}{h(v)} \int_v^{v+uk(v)} h(t) \frac{dt}{k(t)},$$

which for any v is continuous in u; further,

$$g(u) = (1/h(v)) \cdot (h(v + u\theta k(v))/k(v + u\theta k(v))),$$

for some $\theta = \theta_v(u)$ with $0 < \theta_v(u) < 1$. Assuming $h(0) \neq 0$, taking limits as $v \to 0+$ we have for some $0 \leqslant \theta_0(u) \leqslant 1$ that

$$g(u) = (1/h(0)) \cdot (h(u\theta_0 k(0))/k(u\theta_0 k(0))).$$

Now take limits as $u \to 0$: as $g(0+) = 0$,

$$0 = g(0) = 1/k(0) > 0,$$

a contradiction. So $h(0) = 0$. □

Taking $h = k$ so that $p = 1$, which is already continuous, we obtain a corollary which needs only local boundedness above and away from 0, rather than continuity in k (to justify the use of the Beck sequence).

Theorem 10. *Consider the functional equation (GBE) above with $k > 0$ locally bounded above and away from 0 on $\mathbb{R}_+$, and $g(u) \neq 0$ for $u \neq 0$ near 0.*

Any solution is linear: $K(x) = cx$; furthermore, $g(u) = cu$. In particular, for $K = k$ and $g = k - 1$, the solution of the Beurling equation is $k(u) = 1 + cu$.

*Proof***.** This is a simpler version of the preceding proof. With the notation there and with $h = k$ (so that $p = 1$ and $t_{n+1} - t_n = uk(t_n)$), the proof reduces to noting that

$$K(t_m) - K(t_0) = g(u) \sum_{n=0}^{m-1} k(t_n) = \frac{g(u)}{u} \sum_{n=0}^{m-1} uk(t_n) = \frac{g(u)}{u} \sum_{n=0}^{i_1-1} (t_{n+1} - t_n) = \frac{g(u)}{u} t_{i_1},$$

and that for all small enough u we have $g(u)$ non-zero. Then

$$\frac{K(t_{i_1})}{K(t_{i_0})} = \frac{g(u) \sum_{n=0}^{i_1-1} k(t_n)}{g(u) \sum_{n=0}^{m-1} k(t_n)} = \frac{\sum_{n=0}^{i_1-1} uk(t_n)}{\sum_{n=0}^{m-1} uk(t_n)} = \frac{\sum_{n=0}^{i_1-1} t_{n+1} - t_n}{\sum_{n=0}^{i_0-1} t_{n+1} - t_n} = \frac{t_{i_1}}{t_{i_0}},$$

Passing to the limit as $u \to 0$, by continuity of K,

$$K(x_1)/K(x_0) = x_1/x_0 : \qquad K(x) = c_0 x,$$

with $c_0 := K(x_0)/x_0$. $\square$

6 Complements

1. *Regular variation: related results.* Our results here concern the 'Goldie argument', the crux of the remaining 'hard proof' in regular variation (see the proof of Theorem 3 above and [BinO4, §8]). We have focussed particularly here on the key relevant results in BGT, namely Theorem 1.4.3 and Th. 3.2.5, the former simplified in [BinO4, Th. 6], the latter here in Theorem 3. There are a number of related and similar results in BGT Ch. 3. and these may be treated similarly.
2. *Gołąb-Schinzel and related functional equations.* For a recent text-book account of the equation see [AczD, Ch. 19] or the more recent surveys [Brz3] or [Jab], which include generalizations and a discussion of applications in algebra, meteorology and fluid mechanics – see for instance [KahM].
3. *Symmetrized Goldie functional equations.* The equation

$$K(x+y) = g(x)K(y) + g(y)K(x)$$

is studied in [AczD, Ch. 13] in connection with trigonometric identities; interpreting the right-hand side as a discrete convolution with g positive and $g(x)+g(y) = 1$ here also connects to the context of Markov chains – for which see [AczD], Ch. 12]. The 'Goldie case' $g(x) = e^{-\rho x}/2$ yields $K(x) = e^{-\rho x}$.
4. *Differentiability from continuity.* Various equations are known to confer additional regularity on solutions. Here the classical example is the Euler-Lagrange equation; another is provided by d'Alembert's wave equation, for which see [AczD, Ch. 14].
5. *Flows.* The integrator $du/k(u)$ in Theorem 9 above is connected to the *flow* aspects of Beurling regular variation, for which see [BinO3], [Ost1]. The background involves topological dynamics, inspired by Beck [Bec, 5.41]. It also involves regular variation in more general contexts than $\mathbb{R}$, such as normed groups; see [BinO1] and the references cited there for detail.

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