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Beurling moving averages and approximate homomorphisms
by
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Abstract. The theory of regular variation, in its Karamata and Bojanić-Karamata/de Haan forms, is long established and makes essential use of homomorphisms. Both forms are subsumed within the recent theory of Beurling regular variation, developed further here, especially certain moving averages occurring there. Extensive use of group structures leads to an algebraicization not previously encountered here, and to the approximate homomorphisms of the title. Dichotomy results are obtained: things are either very nice or very nasty. Quantifier weakening is extended, and the degradation resulting from working with limsup and liminf, rather than assuming limits exist, is studied.

Key words: Beurling regular variation, Beurling's functional equation, self-neglecting functions, self-equivarying functions, circle group, uniform convergence theorem, category-measure duality, Gołąb-Schinzel functional equation.

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CONTENTS

1 Introduction

This work is a sequel to our recent papers [BinO10,11,12] together with the related papers [Ost2,3,4] by the second author and one [Bin] by the first, re-examined in the light of two much earlier works [BinG2,3] by the first author and Goldie. Our title Beurling moving averages addresses both the Beurling slow and regular variation in [BinO10] (to which we refer for background), and [BinG2,3], the motivation for which is strong laws of large numbers in probability theory.

Beurling regular variation is closely linked with Karamata regular variation (the standard work on which is [BinGT], BGT below, to which we refer for background). In [BinO10], it emerged that Beurling regular variation in fact subsumes the traditional (and very widely used) Karamata regular variation, together with its Bojanić-Karamata/de Haan relative – BGT Ch. 1-3; [BojK], [dH]. Whereas the traditional approach is to develop the measure and Baire-property ('Baire' below) cases in parallel, measure being regarded as primary, it is now clear both that one can subsume both cases together and that it is in fact the Baire case that is primary. This is the theory of topological regular variation, for which see [BinO1,2,4,5], [Ost1] – this informs our approach in §10.

It is convenient to work both multiplicatively in $\mathbb{R}_+ := (0, \infty)$ and additively in $\mathbb{R}$. A self-map f of $\mathbb{R}_+$ or h of $\mathbb{R}$ is *Beurling φ-slowly varying* if, according to context,

$$f(x + t\varphi(x))/f(x) \to 1, \text{ or } h(x + u\varphi(x)) - h(x) \to 0, \qquad (BSV/BSV_+)$$

as $x \to \infty$, where φ is a self-map of $\mathbb{R}_+$ and is *self-neglecting* ($\varphi \in SN$), so that

$$\varphi(x + t\varphi(x))/\varphi(x) \to 1 \text{ locally uniformly in } t \text{ for all } t \in \mathbb{R}_+, \qquad (SN)$$

and $\varphi(x) = o(x)$. This traditional restriction may be usefully relaxed in two ways, as in [Ost3]: firstly, in imposing the weaker order condition $\varphi(x) = O(x)$, and secondly by replacing the limit 1 by a general limit function $\eta > 0$, so that for $\mathbb{A} = [0, \infty)$

$$\eta_x^{\varphi}(t) := \varphi(x + t\varphi(x))/\varphi(x) \to \eta(t) > 0 \text{ locally uniformly in } t \text{ for } t \in \mathbb{A}. \qquad (SE_{\mathbb{A}})$$

Following [Ost3], such a φ will be called *self-equivarying*, $\varphi \in SE$, and the limit function[1] $\eta = \eta^{\varphi}$ necessarily satisfies the *Beurling functional equation*

$$\eta(u + v\eta(u)) = \eta(u)\eta(v) \text{ for } u, v \in \mathbb{R}_+ \qquad (BFE)$$

(this is a special case of the *Gołąb-Schinzel equation* (GS), here conditioned by its relation to $(SE_{\mathbb{A}})$ – see also e.g. [Brz1], [BrzM], or [BinO11]). As $\eta \geqslant 0$, imposing the natural condition $\eta > 0$ (on $\mathbb{R}_+$) above implies that it is continuous and of the form

$$\eta(t) \equiv \eta_\rho(t) := 1 + \rho t, \quad (t \geqslant 0) \qquad \text{for some } \rho \geqslant 0$$

(see [BinO11]). Then we call η a *Beurling function:* $\eta \in GS$, with ρ the η-*index* (of φ when $\eta = \eta^{\varphi}$, and then we write $\rho = \rho_\varphi$); as in BGT 2.11, we extend in §5 the domain (and local uniformity in $(SE_{\mathbb{A}})$) to $\mathbb{A} = (\rho^*, \infty)$, where $\rho^* := -\rho^{-1}$; in §3 we call ρ^* the *Popa origin.* The case $\rho = 0$ recovers SN. For $\varphi \in SE$, a self-map f of $\mathbb{R}_+$ or h of $\mathbb{R}$ is *Beurling φ-regularly varying* if, according to context, the limits below exist:

$$f(x + t\varphi(x))/f(x) \to g(t), \text{ or } h(x + u\varphi(x)) - h(x) \to k(u). \ (BRV/BRV_+)$$

For $\varphi \in SN$ and f Baire/measurable, the limit $g(t)$ is necessarily an exponential function $e^{\gamma t}$ (provided $g > 0$ on a non-negligible set), equivalently k is linear: $k(u) \equiv \gamma u$, convergence is locally uniform, and there is a representation for the possible f (see [BinO10]), involving the *φ-index of Beurling variation,* or *Beurling φ-index* for short, γ. For $\varphi \in SE$ with η-index $\rho > 0$, the situation is altered from $g(t) = e^{\gamma t}$ so that (see [Ost3, Th. 1′])

$$g(t) = (1 + \rho t)^{\gamma}, \text{ or } k(t) = \gamma \log(1 + \rho t) \qquad (t > \rho^*). \qquad (\rho\text{-}BR_\gamma)$$

We are led to the question of existence and additivity properties of the limit functions below:

$$K_F(t) := \lim_{x\to\infty} \Delta_t^{\varphi} F(x)/\varphi(x), \ K_F^*(t) := \limsup_{x\to\infty} \Delta_t^{\varphi} F(x)/\varphi(x),$$

[1] Note the changes here: positivity has been incorporated into the definition (for simplicity), η^{φ} replaces the original notation λ_φ for this context, both to free up the use of λ for other conventional uses, and to reflect the connection to the function H_ρ below (as H denotes the Greek capital 'eta'). Finally, $t = 0$ is included under $(SE_{\mathbb{A}})$ above, being a consequence of the assertion for $t > 0$ – see §5 Lemma 1, Theorem 3.

with Δ_t^φ the difference operator

$$\Delta_t^\varphi F(x) := F(x + t\varphi(x)) - F(x),$$

and local uniform convergence assumed (unless otherwise stated). For $\varphi(x) \equiv 1$ this reduces to the usual difference operator Δ_t. Motivated by classical analysis, we introduce a more general auxiliary function $\psi(x)$ in the denominator:

$$K_F(t) := \lim \Delta_t^\varphi F(x)/\,\psi(x),\;\; K_F^*(t) := \limsup \Delta_t^\varphi F(x)/\,\psi(x).$$

If K_F is defined at u and v, then (cf. §8 Lemma 3)

$$K_F(v + uh(v)) = K_F(v) + K_F(u)g(v),$$

provided

$$h(v) := \lim \varphi(x + v\varphi(x))/\,\varphi(x) \text{ and } g(u) := \lim \psi(x + u\varphi(x))/\,\psi(x)$$

exist (and convergence to K_F is locally uniform), which will be the case when $\varphi \in SE$ (so that $h = \eta_\rho$) and ψ is φ-regularly varying (so that either $\rho = 0$ and $g = e^{\gamma\cdot}$, or $\rho > 0$ and $g \equiv (1 + \rho\cdot)^\gamma$, by ($\rho$ -BR_γ) above). The related functional equation – the extended Goldie-Beurling (Pexiderized[2]) equation,

$$K(v + uh(v)) = K(v) + \kappa(u)g(v), \qquad (GBE\text{-}P)$$

for h, κ positive – is studied in [BinO11, Th. 9 and 10]; special cases appear below in §2 Cor. 2, §8 Lemma 3, §9 Prop. 10. Its solutions K, necessarily continuous, are there characterized (subject to $K(0) = 0$) as

$$K(x) \equiv c \cdot \tau_f(x) \text{ with } f := h/g \text{ and } \tau_f(x) := \int_0^x \mathrm{d}w/f(w) \qquad (x \geqslant 0),$$

an 'occupation time measure' (of the interval $[0, x]$; §2) and $c \in \mathbb{R}$; the 'relative flow rate' f satisfies the *Cauchy-Beurling exponential equation*:

$$f(v \circ_h u) = f(u)f(v), \qquad (CBE)$$

cf. [Chu], [Ost4]. Here $\circ_h$ denotes Popa's binary operation ([Pop], cf. [Jav], §3 below)

$$v \circ_h u := v + uh(v),$$

[2] After Pexider's equation: $f(xy) = g(x) + h(y)$ in three unknown functions and its generalizations – cf. [Kuc, 13.3], [Brz1, 2]. See also [Ste] for the more general Levi-Civita functional equations.

so that $h = \eta_\rho$ itself also satisfies (CBE); this confers a group structure, turning certain subsets of $\mathbb{R}$ into groups, called *Popa (circle) groups* in §3; furthermore, necessarily $\kappa = K$. Solving $(GBE\text{-}P)$ may be expressed as an equivalent Popa *homomorphism problem* of finding $k, h \in GS$ satisfying

$$K(v \circ_h u) = K(v) \circ_k K(u) \tag{GBE}$$

(cf. [Brz2], [Mur], [Ost4]), where

$$k(u) = g(K^{-1}(u)).$$

This observation is new even for the classical context $h \equiv 1$; here $f = e^{-\gamma t}$, so

$$\tau_f(x) \equiv H_\gamma(x) := (e^{\gamma x} - 1)/\gamma \text{ with } H_0(x) \equiv x.$$

For $\eta \equiv \eta_\rho$ with $\rho > 0$, $g \equiv (1 + \rho\cdot)^\gamma$, by $(\rho\text{ -}BR_\gamma)$ above, $f(x) = (1 + \rho x)^{1-\gamma}$, so for $x > \rho^*$,

$$K \equiv c\cdot\tau_f = c\cdot K_{\rho\gamma}, \text{ where } K_{\rho\gamma}(x) := \int_0^x (1+\rho w)^{\gamma-1} \mathrm{d}w = ((1 + \rho x)^\gamma - 1)/\rho\gamma$$

(linear for $\gamma = 1$). The 'slow case' $\gamma = 0$ may also be handled via

$$\lim_{\gamma \to 0} K_{\rho\gamma}(x) = \log(1 + \rho x)/\rho \qquad (x > \rho^*).$$

When $\varphi(x) \equiv 1$, the moving averages $\Delta_t^\varphi F(x)/\psi(x)$ reduce to classical Bojanić-Karamata/de Haan limits (BGT Ch. 3), for which the auxiliary $\psi(x)$ is necessarily Karamata regularly varying, so just as before (trivially, since $\varphi \in SE$) has exponential limit function, $g \equiv e^{\gamma\cdot}$ say, and then $(GBE\text{-}P)$ simplifies to the original *Goldie functional equation* (see e.g. [BinO11,12], [Ost4]):

$$K(u + v) = e^{\gamma u} K(v) + K(u), \tag{GFE}$$

with solution $K(u) \equiv c \cdot H_\gamma(u)$, as before. The latter function plays a crucial role in the Bojanić-Karamata/de Haan theory of regular variation. Here, and in the general case, if $\Delta_t^\varphi F/\psi$ has a limiting moving average K_F, then for some $c_F \in \mathbb{R}$, as above (cf. [BinO11, Th. 3, 9, 10]),

$$K_F(u) = c_F \cdot H_\gamma(u),$$

with c_F the *ψ-index* of F (for ψ which is φ-regularly varying), while ψ has *Beurling φ-index* γ.

In the classical context, with limsup in place of limit one works also with K_F^*, abbreviated to K^* (and similarly K_* with liminf). Here the equations (GFE) give way to functional inequalities, such as the *Goldie functional inequality*

$$K^*(u+v) \leqslant e^{\gamma u} K^*(v) + K^*(u) \qquad (GFI)$$

(BGT (3.2.5)), which we summarize by saying that K^* is *exp-subadditive.* Equivalently, this may be re-expressed symmetrically here as *group subadditivity*:

$$K^*(x+y) \leqslant K^*(x) \circ_k K^*(y)$$

with k as above, and in the more general Beurling case correspondingly to (GBE) as

$$K^*(x \circ_h y) \leqslant K^*(x) \circ_k K^*(y).$$

For ψ regularly varying, the set

$$\mathbb{A} := \{t : \lim \Delta_t F(x)/\psi(x) \text{ exists and is finite}\},$$

for which see e.g. BGT Th. 3.2.5 (proof) and §§5,6 below, constitutes the domain of the function

$$K_F(a) := \lim_{x\to\infty} \Delta_a F(x)/\psi(x) \qquad (a \in \mathbb{A}); \qquad \text{(ker)}$$

hence we refer to K_F here and above as the *regular kernel* of F – the homomorphism approximating F of our title. In [BinO11] (and in [BinO12] for the case $\rho = 0$), we study conditions on K^* implying that K_F exists, i.e. that the inequality becomes an equation, by imposing 'Heiberg-Seneta' side-conditions (see §7 Prop. 9), and density of $\mathbb{A}$ – again cf. BGT Ch. 3, especially the crucial Theorem 4.2.5. Below these findings are extended to the Beurling context.

In view of the algebraic treatment to follow in §3 on Popa groups, one may regard the terms *additive* and *homomorphic* as synonymous for our purposes here.

2 From Beurling to Karamata

The function H_ρ (of §1) satisfies

$$dH_\rho/\,dx = e^{\rho x} = 1 + \rho H_\rho(x) = \eta_\rho(H_\rho(x)),$$

and solves the Goldie equation (GFE), in which the auxiliary function g, which is necessarily exponential for K Baire/measurable, takes the form $g(x) = e^{\rho x}$ – again see [BinO11, Th. 1]. Regarding $\varphi, \eta \in SE$ as generating (velocity) flows as in [BinO10], their occupation 'times' (on $[0, x]$) are (cf. [Bec, p.153]):

$$\tau_\varphi(x) := \int_0^x \mathrm{d}w/\varphi(w) \text{ and } \tau_\eta(x) := \int_0^x \mathrm{d}w/\eta(w),$$

both strictly increasing. (For present needs this notation is more symmetrical than that of [BinG1] with Φ for τ_φ, and of BGT 2.12.29, which we mention for purposes of comparison.) For $\rho > 0$ and $\eta = \eta_\rho \in SE$

$$\tau_\eta(x) := \int_0^x \frac{\mathrm{d}w}{1 + \rho w} = \frac{1}{\rho} \log(1 + \rho x),$$

so

$$\tau_\eta^{-1}(t) = H_\rho(t) = (e^{\rho t} - 1)/\rho.$$

In particular, the trajectory $w(t) := \tau_\eta^{-1}(t)$ satisfies the equation

$$\mathrm{d}w(t)/\mathrm{d}t = e^{\rho t} = 1 + \rho w(t) = \eta(w(t)) \text{ with } w(0) = 0.$$

Necessarily, working with the (inverse) re-parametrization $\mathrm{d}t(w)/\mathrm{d}w = e^{-\rho t} = \psi(t) \in SE$ gives $\tau_\psi(x) = H_\rho(x)$, again an occupation time measure.

We now generalize a theorem of Bingham and Goldie [BinG2, Th. 2]. This recovers their theorem when $\rho_\eta = 0$ and $\varphi(x) = o(x)$, as then $\varphi \in SN$. The result may be interpreted as a local 'chain rule', for $V(s) = U(s(t))$, where the trajectory $s(t) := \tau_\varphi^{-1}(t)$ satisfies $\mathrm{d}s(t)/\mathrm{d}t = \varphi(s(t)) = \varphi(\tau_\varphi^{-1}(t)) = g(t)$ (with $\varphi \in SE$, a 'self-equivarying flow').

Theorem 1 (Time-change Equivalence Theorem for Moving Averages). *For positive $\varphi \in SE$ with $1/\varphi$ locally integrable, U satisfies*

$$\frac{U(x + t\varphi(x)) - U(x)}{\varphi(x)} \to c_U t \text{ as } x \to \infty, \text{ for all } t \geqslant 0 \qquad (BMA_\varphi)$$

*iff its **time-changed version** $V := U \circ \tau_\varphi^{-1}$ satisfies, for $g(y) := \varphi(\tau_\varphi^{-1}(y))$,*

$$\frac{V(y + s) - V(y)}{g(y)} \to c_U H_\rho(s) \text{ as } y \to \infty, \text{ for all } s \geqslant 0, \qquad (KMA_g)$$

where $\rho = \rho_\varphi$ *is the* η*-index of* φ.

This is proved exactly as in [BinG2, Th. 2], using the following.

Proposition 1. *For* $\varphi \in SE$ *and* $\eta = \eta^\varphi$, *locally uniformly in* s

$$\lim[\tau_\varphi(x + s\varphi(x)) - \tau_\varphi(x)] = \tau_\eta(s).$$

In particular, this is so for $\varphi \in SN$, *where* $\tau_\eta(s) \equiv s$.

Proof. Let ρ be the η-index. Fix $s > 0$, then uniformly in $t \in [0, s]$

$$\varepsilon(x,t) := \varphi(x)/\varphi(x + t\varphi(x)) - 1/\eta(t) \to 0, \text{ so } e(x,s) := \int_0^s \varepsilon(x,t)\mathrm{d}t \to 0.$$

Then, as in [BinG2, Th. 2], using the substitution $w = x + t\varphi(x)$

$$\begin{aligned} \tau_\varphi(x + s\varphi(x)) - \tau_\varphi(x) &= \int_x^{x+s\varphi(x)} \mathrm{d}w/\varphi(w) = \int_0^s \frac{\varphi(x)\mathrm{d}t}{\varphi(x + t\varphi(x))} \\ &= \int_0^s \left(\frac{1}{\eta(t)} + \varepsilon(x,t)\right)\mathrm{d}t = \tau_\eta(s) + e(x,u). \end{aligned}$$

If $\varphi \in SN$, then $\tau_\eta(s) \equiv s$, as $\eta \equiv 1$. $\square$

Our first corollary characterizes SE in terms of a *multiplicative* Karamata index via its time-changed version g; this is a *consistency* result in view of the characterization from [Ost3] of $\varphi \in SE$ as the product $\eta^\varphi\psi$ with ψ in SN. The latter identifies φ itself as having *additive* Karamata index ρ_φ.

Corollary 1. $\varphi \in SE$ *iff* $g = \varphi \circ \tau_\varphi^{-1}$ *is regularly varying in the additive-argument sense with multiplicative Karamata index* ρ_φ. *In particular,* $\varphi \in SN$ *iff* $g = \varphi \circ \tau_\varphi^{-1}$ *is regularly varying with multiplicative Karamata index* $\rho_\varphi = 0$.

Proof. Put $\rho = \rho_\varphi$. Since

$$(\varphi(x + t\varphi(x)) - \varphi(x))/\,\varphi(x) = \varphi(x + t\varphi(x))/\,\varphi(x) - 1 \to \rho t,$$

we may apply Th. 0 to $U = \varphi$ so that $V := \varphi \circ \tau_\varphi^{-1} = g$; then by (KMA_g)

$$g(y+s)/\,g(y) - 1 = (g(y+s) - g(y))/\,g(y) \to (e^{\rho_\varphi x} - 1): \quad g(y+s)/\,g(y) \to e^{\rho x},$$

and conversely. $\square$

If $K_V(s)$ – defined by (ker) above (with g for ψ) – exists for all s, as in (KMA_g), then as we now show K_V satisfies a Goldie equation, from which the form of K_V can be read off, as in the Equivalence Theorem, Theorem 1.

Corollary 2. *For $\varphi \in SE$, so that $g = \varphi \circ \tau_\varphi^{-1}$ is regularly varying with multiplicative Karamata index $\rho = \rho_\varphi$:*
if KMA_g – equivalently BMA_φ – holds, then for $K_V(u)$, as above,

$$K_V(s+t) = K_V(s)e^{\rho t} + K_V(t),$$

and so for some c

$$K_V(s) = cH_\rho(s).$$

Proof. The Goldie equation follows from Corollary 1, since

$$\frac{V(y+s+t)-V(y)}{g(y)} = \frac{V(y+s+t)-V(y+t)}{g(y+t)}\frac{g(y+t)}{g(y)} + \frac{V(y+t)-V(y)}{g(y)}.$$

Now apply Theorem 2 of [BinO11] to deduce the form of K_V. $\square$

3 Popa (circle) groups

Recall from Popa [Pop], for $h : \mathbb{R} \to \mathbb{R}$, the *Popa operation* $\circ_h$ and its *Popa domain* $\mathbb{G}_h$ (our terminology) defined by:

$$a \circ_h b := a + bh(a), \qquad \mathbb{G}_h := \{g : h(g) \neq 0\}.$$

The special case (but nevertheless typical – see below) of $h(t) = \eta_1(t) \equiv 1+t$ yields the *circle product* in a ring, $a \circ b := a+b+ab$ – see [Ost4] for background. We recall also, from Javor [Jav] (in the broader context of $h : \mathbb{E} \to \mathbb{F}$, with $\mathbb{E}$ a vector space over a commutative field $\mathbb{F}$), that $\circ_h$ is associative iff h satisfies the Gołąb-Schinzel equation, briefly $h \in GS$ (cf. §1 – a temporary ambiguity resolved below):

$$h(x+yh(x)) = h(x)h(y) \qquad (x, y \in \mathbb{G}_h). \tag{GS}$$

Their role below is fundamental; first, $GS \subseteq SE$, and for $\varphi \in SE$ the Popa operation $x \circ_\varphi t = x + t\varphi(x)$ compactly expresses the Beurling transformation

$t \to x + t\varphi(x)$. More is true: taking one step further beyond GS to SE is an operation localized to x :

$$s \circ_{\varphi x} t := s + t\eta_x^{\varphi}(s), \text{ where } \eta_x^{\varphi}(s), \text{ or just } \eta_x(s) := \varphi(x + s\varphi(x))/\varphi(x)$$

as in §1 (we use η_x^{φ} or η_x depending on emphasis or context). The notation above neatly summarizes two frequently used facts in (Karamata/Beurling) regular variation:

$$x \circ_{\varphi} (b \circ_{\varphi x} a) = y \circ_{\varphi} a, \text{ for } y = x + b\varphi(x)$$

(proved in Prop. 2(ii) below), and as $x \to \infty$, locally uniformly in s, t :

$$s \circ_{\varphi x} t \to s \circ_{\eta} t, \text{ for } \eta(s) := \lim_x \eta_x^{\varphi}(s) \in GS.$$

So here we return to GS.

The appearance of a group structure 'in the limit' is not accidental – see [Ost4] for background. The fact that, for η as here, $\eta \in GS$ is proved in [Ost3] – see §1; solutions of (GS) that are *positive* on $\mathbb{R}_+ := (0, \infty)$ are key here, being of the form $\eta_{\rho}(x) := 1 + \rho x$ with $\rho \geqslant 0$. The case $\rho = 0$ corresponds to the classical Karamata setting, and $\rho > 0$ to the recently established, general, theory of Beurling regular variation [BinO10]. For the corresponding *Popa groups* write $\circ_{\rho}$ (when $h = \eta_{\rho}$), or even $\circ$, omitting subscripts both on $\circ$ and on η, if context permits. To prevent confusion, $u_{\circ}^{-1}$ denotes the relevant group inverse. Furthermore, we employ the notation:

$$\rho^* := -\rho^{-1}, \quad \mathbb{G}_*^{\rho} := \mathbb{R}\backslash\{\rho^*\}, \quad \mathbb{G}_+^{\rho} := (\rho^*, \infty), \quad \mathbb{G}_-^{\rho} := (-\infty, \rho^*), \qquad (\rho \neq 0),$$

$$\mathbb{G}_*^{\infty} := \mathbb{R}^* = \mathbb{R}\backslash\{0\}, \qquad \mathbb{G}_*^0 := \mathbb{R},$$
$$\eta_*^{\rho}(x) := \eta_{\rho}(x) \ (\rho \neq 0), \quad \eta_*^0(x) := e^x.$$

We call ρ^* the *Popa origin* (of $\mathbb{G}_*^{\rho}$), interpreting it when $\rho = 0$ as $-\infty$. Its critical role for Beurling regular variation emerges in §5 Lemma 1.

We collect relevant facts in the following, slightly extending work of Popa [Pop, Prop. 2] and Javor [Jav, Lemma 1.2].

Theorem PJ. *For φ satisfying (GS) above, not the zero map, $(\mathbb{G}_{\varphi}, \circ_{\varphi})$ is a group. If φ is injective on $\mathbb{G}_{\varphi}$, then $\circ_{\varphi}$ is commutative, and φ maps homomorphically into $(\mathbb{R}^*, \cdot)$:*

$$\varphi(x \circ_{\varphi} y) = \varphi(x)\varphi(y).$$

In particular, $\mathbb{G} = \mathbb{G}^\rho := (\mathbb{G}^\rho_*, \circ_\rho)$ *is an abelian group with* $1_\mathbb{G} = 0$ *and inverse*

$$u_\circ^{-1} = -u/\eta_\rho(u).$$

$\mathbb{G}^0 := (\mathbb{R}, \circ)$ *is* $(\mathbb{R}, +)$ *for* $\rho = 0$*, so that* $\mathbb{G}^\rho$ *is isomorphic under* η^ρ_* *to* $(\mathbb{R}^*, \cdot)$ *for* $\rho \geqslant 0$*. Furthermore,*
(i) *inversion carries* $\mathbb{G}^\rho_+$ *into itself:* $(\mathbb{G}^\rho_+)^{-1}_\circ = \mathbb{G}^\rho_+$ *and* η^*_ρ *carries* $\mathbb{G}^\rho_+$ *onto* $\mathbb{R}_+$*;*
(ii) *for* $\rho > 0$ *the reflection* $\pi = \pi_\rho : u \mapsto -u + 2\rho^*$ *carries each of* $\mathbb{G}^\rho_+$ *and* $\mathbb{G}^\rho_-$ *skew-isomorphically onto the other in the sense that*

$$\pi^{-1}(\pi(s) \circ_\rho \pi(t)) = \pi(s \circ_\rho t),$$

and

$$|\eta^\rho_*(\pi(t))| = \eta^\rho_*(t) \qquad (t \in \mathbb{G}^\rho_+); \qquad \eta^\rho_*(\rho^*) = 0.$$

Proof. In general, if φ is injective on $\mathbb{G}_\varphi$, then $\circ_\varphi$ is commutative, as (GS) is symmetric on the right-hand side. Commutativity of $\circ_\rho$ follows directly from $v + u(1 + \rho v) = u + v(1 + \rho u)$. As $u \circ 0 = u$ and $0 \circ v = v$, the neutral element is $1_\mathbb{G} = 0$; the inverse is

$$v_\circ^{-1} = -v/\eta(v) = -v/(1 + \rho v) \text{ for } x \in \mathbb{G}_\rho \text{ (as } v \neq \rho^*).$$

Isomorphic maps of $\mathbb{G}$ are provided for $\rho = 0$ by $\iota : x \mapsto x$ onto $(\mathbb{R}, +)$, and for $\rho > 0$ by $\eta : x \to 1 + \rho x$ onto $(\mathbb{R}_+, \cdot)$, since

$$\eta(u)\eta(v) = (1 + \rho u)(1 + \rho v) = 1 + \rho[u + v(1 + \rho u)] = \eta(u \circ_\eta v).$$

The rest follows since $\rho > 0$ and $x > -1/\rho$ imply $\eta(x) > 0$. Also, as $\rho\rho^* = -1$,

$$\begin{aligned} (2\rho^* - s) + (2\rho^* - t)(1 + \rho(2\rho^* - s)) &= 4\rho^* - s - t + (2\rho^* - t)(-2 - \rho s)) \\ &= s + t(1 + \rho s) = s \circ_\rho t \\ &= \pi^2(s \circ_\rho t), \end{aligned}$$

(as $\pi^2 = \iota$) and $|\eta^\rho_*(\pi(t))| = |1 + \rho[-t - 2/\rho]| = |-1 - \rho t| = \eta^\rho_*(t)$, for $t \in (\rho^*, \infty)$. $\square$

Remarks. 1. For $\rho \neq 0$, $\mathbb{G}^\rho$ is typified (rescaling its domain) by the case $\rho = 1$, where

$$a \circ_1 b = (1 + a)(1 + b) - 1 : \qquad (\mathbb{G}^1, \circ_1) = (\mathbb{R}^*, \cdot) - 1,$$

and the isomorphism is a shift (cf. [Pop, §3]), i.e. the groups are conjugate. This is the classical *circle group* above.
2. For $\rho > 0$, note that $u \in \mathbb{G}_+^\rho \cap (0, \infty)$ has $u_\circ^{-1} \in \mathbb{G}_+^\rho \cap (-1/\rho, 0)$.
3. Since $\eta(t_\circ^{-1}) = 1/\eta(t)$, $t_\circ^{-1} \circ v = (v-t)/\eta(t)$, and so the convolution $t * v := v \circ t_\circ^{-1}$ is the asymptotic form of the Beurling convolution $(v-t)/\varphi(t)$ occurring in the Beurling Tauberian Theorem (§4) for $\varphi \in SN$.
4. For $\rho > 0$, the inverse $\eta^{-1}(y) = (y-1)/\rho$ maps $(0, \infty)$ onto $\mathbb{G}$; moreover, η^{-1} is *super-additive* on $(1, \infty)$, i.e. for $x, y \geqslant 1 = 1_{\mathbb{R}^*}$,

$$\eta^{-1}(x) + \eta^{-1}(y) \leqslant \eta^{-1}(xy),$$

as

$$0 \leqslant \rho^2 \eta^{-1}(x)\eta^{-1}(y) = (xy-1)-(x-1)-(y-1) = \rho\eta^{-1}(xy)-\rho\eta^{-1}(x)-\rho\eta^{-1}(y);$$

it is also super-additive on $(0, 1)$.

Below we list further useful arithmetic facts including the iterates $a_{\varphi x}^{n+1} = a_{\varphi x}^n \circ_{\varphi x} a$ with $a_{\varphi x}^1 = a$. To avoid excessive bracketing, the usual arithmetic operations below bind more strongly than Popa operations.

Proposition 2 (Arithmetic of Popa operations).

i) $a_{\varphi x}^0 = 1_{\varphi x} = 0; \qquad a \circ_{\varphi x} a_{\varphi x}^{-1} = 0 \qquad \text{for } a_{\varphi x}^{-1} := (-a)/\eta_x^\varphi(a);$

ii) $x \circ_\varphi (b \circ_{\varphi x} a) = y \circ_\varphi a, \qquad \text{for } y := x \circ_\varphi b;$

iii) $x \circ_\varphi (b \circ_\eta a) = y \circ_\varphi a\eta(b)/\eta_x^\varphi(b) \qquad \text{for } y := x \circ_\varphi b;$

iv) $x = y \circ_\varphi b_{\varphi x}^{-1} \qquad \text{for } y := x \circ_\varphi b;$

v) $\eta_x^\varphi(a_{\varphi x}^m) = \prod_{k=0}^{m-1} \eta_{y_k}^\varphi(a), \qquad \text{for the iterates } a_{\varphi x}^n \text{ and } y_k = x \circ_\varphi a_{\varphi x}^k, \ (k = 0, ..., m-1).$

Proof. (i) Here $1_{\varphi x}$ denotes the neutral element of the operation $\circ_{\varphi x}$, which is 0, since $\eta_x^\varphi(0) = 1$ (so that $0 \circ_{\varphi x} t = t$, while $s \circ_{\varphi x} 0 = s$). So $a_{\varphi x}^1 = a_{\varphi x}^0 \circ_{\varphi x} a$.

$$a \circ_{\varphi x} a_{\varphi x}^{-1} = a + a_{\varphi x}^{-1}\eta_x^\varphi(a) = 0.$$

(ii) For $y = x \circ_\varphi b$,

$$x \circ_\varphi (b \circ_{\varphi x} a) = x \circ_\varphi (b + a\eta_x^\varphi(b)) = x + b\varphi(x) + a\varphi(x + b\varphi(x)) = y \circ_\varphi a.$$

(iii) As in the preceding step for (ii),

$$x + (b + a\eta(b))[\varphi(x)/\varphi(x \circ_\varphi b)]\varphi(x \circ_\varphi b) = y \circ_\varphi a\eta(b)/\eta_x^\varphi(b).$$

(iv) For $y = x \circ_\varphi b$, using $b_{\varphi x}^{-1} = -b/\eta_x^\varphi(b)$ from (i),

$$x = y - b\varphi(x) = y - [b\varphi(x)/\varphi(y)]\varphi(y) = y \circ_\varphi b_{\varphi x}^{-1}.$$

(v) For $m = 1$ both sides agree since by (i) $y_0 = x$. Proceed by induction, using (ii):

$$\begin{aligned}\eta_x^\varphi(a_{\varphi x}^{m+1}) &= \varphi(x \circ_\varphi (a_{\varphi x}^m \circ_{\varphi x} a))/\varphi(x) = \varphi(y_m \circ_\varphi a))/\varphi(x) \\ &= [\varphi(y_m \circ_\varphi a)/\varphi(y_m)]\varphi(x \circ_\varphi a_{\varphi x}^m)/\varphi(x) = \eta_{y_m}^\varphi(a)\eta_x^\varphi(a_{\varphi x}^m). \ \square\end{aligned}$$

4 Extension to Beurling's Tauberian Theorem

Theorem 2 below extends one proved by Beurling in lectures in 1957; see e.g. [Kor, IV.11] for references. Bingham and Goldie [BinG2] extended Beurling's result by replacing the Lebesgue integrator $H(y)dy$ below by a suitable Lebesgue-Stieltjes integrator $dU(y)$, and demanding more of the Wiener kernel (than just non-vanishing of its Fourier transform), and gave a corollary for Beurling moving averages.

Here we extend the class of Beurling convolutions applied in the other term of the integrand, replacing $\varphi \in BSV$ by $\varphi \in SE$, so widening the application to moving averages, as we note below. With the following 'Beurling notation' for Lebesgue and Stieltjes integrators,

$$\begin{aligned}F *_\varphi H(x) &:= \int F\Big(\frac{x-u}{\varphi(x)}\Big)H(u)\frac{\mathrm{d}u}{\varphi(x)} = \int F(-t)H(x + t\varphi(x))\mathrm{d}t, \\ F *_\varphi dU(x) &:= \int F\Big(\frac{x-u}{\varphi(x)}\Big)\frac{\mathrm{d}U(u)}{\varphi(x)} = \int F(-t)dU(x + t\varphi(x))\mathrm{d}t,\end{aligned}$$

reducing for $\varphi \equiv 1$ to their classical counterparts

$$F * H(x) = \int F(x-t)H(t)\mathrm{d}t, \qquad F * dU(x) = \int F(x-t)\mathrm{d}U(t),$$

we recall Wiener's theorem for the Lebesgue and the Lebesgue-Stieltjes integrals. The latter uses the class $\mathcal{M}$ of continuous functions (see Widder [Wid, V.12]; cf. [Wie, II.10]) with norm:

$$||f|| := \sup_{y\in\mathbb{R}} \sum_{n\in\mathbb{Z}} \sup_{x\in[0,1]} |f(x + y + n)| < \infty,$$

and places a uniform bounded-variation restriction on the integrator U as follows. Denote by $|\mu_x|$ the usual norm of the charge (signed measure) generated from the function $y \mapsto U_x(x \circ_\varphi y)/\varphi(x)$; then there should exist $\delta > 0$ and $M < \infty$ with

$$\sup_{x,y\in\mathbb{R}} |\mu_x|(I_\delta^+(y)) \leqslant M, \tag{BV}$$

where $I_\delta^+(y) := [y, y+\delta)$. It will be convenient to refer to the following conditions as $x \to \infty$, with or without the subscript φ (the latter when $\varphi \equiv 1$) :

$$K *_\varphi H(x) \to c\int K(y)\mathrm{d}y, \quad K *_\varphi dU(x) \to c\int K(y)\mathrm{d}y. \tag{$K *_\varphi H/U$}$$

Theorem W (Wiener's Tauberian Theorem). *For $K \in L_1(\mathbb{R})$ (resp. $K \in \mathcal{M}$) with $\hat{K}$ non-zero on $\mathbb{R}$:*

if H is bounded (resp. $H \in \mathcal{M}$), and $(K * H)$, resp. $(K * U)$, holds, then for all $F \in L_1(\mathbb{R})$ (resp. $F \in \mathcal{M}$),

$$F * H(x), \text{ resp. } F * \mathrm{d}U(x) \to c\int F(t)\mathrm{d}t \qquad (x \to \infty).$$

Theorem B (Beurling's Tauberian theorem). *For $K \in L_1(\mathbb{R})$ with $\hat{K}$ non-zero on $\mathbb{R}$, and φ Beurling slowly varying,*

$$\varphi(x + t\varphi(x))/\varphi(x) \to 1, \qquad (x \to \infty) \qquad (t \geqslant 0) : \tag{BSV}$$

if H is bounded, and $(K *_\varphi H)$ holds, then for all $F \in L_1(\mathbb{R})$

$$F *_\varphi H(x) \to c\int F(y)\mathrm{d}y \qquad (x \to \infty).$$

We recommend the much later, slick, and elegant proof in [Kor, IV.11].

Theorem BG1 (LS-Extension to Beurling's Tauberian theorem, [BinG2, Th. 8]). *If $\varphi \in BSV$, $K \in \mathcal{M}$ with $\hat{K}$ non-zero on $\mathbb{R}$, U satisfies (BV) and $(K *_\varphi U)$ holds*

— then for all $G \in \mathcal{M}$,

$$G *_\varphi \mathrm{d}U(x) \to c\int G(y)\mathrm{d}y \qquad (x \to \infty).$$

We show how to amend the proof of Th. BG1 in [BinG2] (similar in essence to that cited above in [Kor, IV.11]) to obtain the following.

Theorem 2 (Extension to Beurling's Tauberian theorem). *If $\varphi \in SE$, i.e. locally uniformly in t*

$$\varphi(x+t\varphi(x))/\varphi(x) \to \eta(t) \in GS, \qquad (x \to \infty) \qquad (t \geqslant 0), \qquad (SE)$$

$K \in L_1(\mathbb{R})$ (resp. $K \in \mathcal{M}$) with $\hat{K}$ non-zero on $\mathbb{R}$, H is bounded (resp.U satisfies (BV)) and $(K *_\varphi H)$, resp. $(K *_\varphi U)$, holds
— then for all $G \in L_1(\mathbb{R})$ (resp. $G \in \mathcal{M}$)

$$G *_\varphi H(x) \to c \int G(y)\mathrm{d}y, \qquad \text{resp. } G *_\varphi \mathrm{d}U(x) \to c \int G(y)\mathrm{d}y \qquad (x \to \infty).$$

Proof. We consider the Lebesgue-Stieltjes case (the Lebesgue case is similar, but simpler). For fixed a and with K as in the Theorem, set $K_a(s) := K(s-a)$, and take

$$t := (s-a)/\eta_x(a),\ dt = ds/\eta_x(a) \text{ and } s = a + t\eta_x(a) = a \circ_{\varphi x} t.$$

Then for $y = x + a\varphi(x)$, by Prop. 2(ii), $x \circ_\varphi (a \circ_{\varphi x} t) = y \circ_\varphi a$ and so

$$\begin{aligned} K_a(s)U(x \circ_\varphi s) &= K(t\eta_x(a))U(x \circ_\varphi (a \circ_{\varphi x} t)) \\ &= K(t\eta_x(a))U(y \circ_\varphi t). \end{aligned}$$

So, as in [BinG2], for K continuous ($K \in \mathcal{M}$), with A as in [BinG2] for c above

$$\begin{aligned} \int K_a(s)\mathrm{d}U(x \circ_\varphi s) &= \eta_x(a) \int K(t\eta_x(a))\mathrm{d}U(y \circ_\varphi t) \to A \int K(t\eta(a))\eta(a)\mathrm{d}t \\ &= A \int K(u)\mathrm{d}u, \text{ for } u := t\eta(a). \end{aligned}$$

Now continue with the proof verbatim as in [BinG2]. □

Corollary 3 ([BinG2, §5 Cor. 2] for $\varphi \in SN$). *For $\varphi \in SE$, if U is non-decreasing and for some $\delta > 0$*

$$\sup_{x,y \in \mathbb{R}} [U_x(x \circ_\varphi (y+\delta)) - U_x(x \circ_\varphi y]/\varphi(x) < \infty$$

— *then* $(K *_\varphi U)$ *holds for some* c *and Wiener kernel* $K \in \mathcal{M}$ *iff for some* c_U *either of the following holds:*

$$(\Delta_t^\varphi U/\varphi)(x) \equiv [U(x \circ_\varphi t) - U(x)]/\varphi(x) \to c_U t \qquad (x \to \infty) \qquad (t > 0),$$

$$(\Delta_t^\varphi U/\varphi)(x) \to c_U t \qquad (x \to \infty) \qquad \text{for two incommensurable } t.$$

Proof. Repeat verbatim the proof in [BinG2, §5 Cor. 2], using $H(x) = t^{-1}\mathbf{1}_{[0,t]}(x)$, with $\mathbf{1}_{[0,t]}$ the indicator function of the interval $[0,t]$. $\square$

5 Uniformity, semicontinuity

To motivate our results below of limsup convergence type, we use the following weak notion of uniformity: say that $f_n \to f$ *uniformly near* t if for every $\varepsilon > 0$ there is $\delta > 0$ and $m \in \mathbb{N}$ such that

$$f(t) - \varepsilon < f_n(s) < f(t) + \varepsilon \text{ for } n > m \text{ and } s \in I_\delta(t),$$

where $I_\delta(t) := (t-\delta, t+\delta)$. For instance, if $\varphi \in SN$, x_n divergent, $f(s) \equiv 1$, and $f_n(s) := \varphi(x_n + s\varphi(x_n))/\varphi(x_n)$, then '$f_n \to f$ uniformly near t for all $t > 0$.'

The notion above is easier to satisfy than Hobson's 'uniform convergence *at* t' which replaces $f(t)$ above by $f(s)$ twice, [Hob, p.110]; suffice it to refer to $f_n \equiv 0$, and f with $f(0) = 0$ and $f \equiv 1$ elsewhere. (See also Klippert and Williams [KliW], where though Hobson's condition is satisfied at all points of a set, the choice of δ cannot itself be uniform in t.)

Our notion of uniformity may be equivalently stated in limsup language, as follows, bringing to the fore the underlying *uniform upper and lower semicontinuity*. The proof of the next result is routine, so we omit it here[3]; but its statement will be useful in the development below.

Proposition 3 (Uniform semicontinuity). *If* $f_n \to f$ *pointwise, then* $f_n \to f$ *converges uniformly near* t *iff*

$$\begin{aligned} f(t) &= \lim_{\delta \downarrow 0} \limsup_n \sup\{f_n(s) : s \in I_\delta(t)\} \\ &= \lim_{\delta \downarrow 0} \liminf_n \inf\{f_n(s) : s \in I_\delta(t)\}. \end{aligned}$$

[3]See the Appendix for details.

Again putting $I_\delta^+(t) := [t, t+\delta)$, we may now consider the *one-sided limsup-sup condition* at t :

$$f(t) = f_+(t) \text{ with } f_+(t) := \lim_{\delta\downarrow 0} \limsup_n \sup\{f_n(s) : s \in I_\delta^+(t)\}. \qquad (1)$$

The next result is akin to the Dini/Pólya-Szegő monotone convergence theorems (respectively [Rud,7.13], for monotone convergence of continuous functions to a continuous pointwise limit, and [PolS, Vol. 1 p.63, 225, Problems II 126, 127], or Boas [Boa, §17, p. 104-5], when the functions are monotone); here we start with one-sided assumptions on the domain and range, and conclude via a category argument by improving to a two-sided condition.

Proposition 4 (Uniform Upper semicontinuity). *If quasi everywhere f_n converges pointwise to an upper semicontinuous limit f satisfying the one-sided condition* (1) *quasi everywhere in its domain, then quasi everywhere f is uniformly upper semicontinuous:*

$$f(t) = \lim_{\delta\downarrow 0} \limsup_n \sup\{f_n(s) : s \in I_\delta(t)\}.$$

Proof. Take $\{I_n\}_{n\in\mathbb{N}}$ with $I_0 = \mathbb{R}$ to be a sequence of open intervals that form a base for the usual open sets of $\mathbb{R}$. Let $\mathbb{D}$ be a countable dense subset of the (co-meagre) intersection S of the set on which f_n converges pointwise and the set on which (1) holds. Put

$$G^k(\varepsilon) := \bigcup\nolimits_{m\in\mathbb{N}} \{(q, q+\delta) : q \in \mathbb{D},\ (q, q+\delta) \subseteq I_k,\ (\forall n > m)(\forall s \in I_\delta^+(q)) \\ [f_n(s) < f(q) + \varepsilon]\},$$

which is open. It is also dense in I_k: from any open interval $I \subseteq I_k$ choose $q \in \mathbb{D}\cap I$; as $q \in S \cap I$, there exist $N_k \in \mathbb{N}$ and $\delta > 0$ such that $I_\delta^+(q) \subseteq I$ and

$$f_n(s) < f(q) + \varepsilon \qquad (n > N_q,\ s \in I_\delta^+(q));$$

so $(q, q+\delta) \subseteq I \cap G^k(\varepsilon)$, i.e. $G^k(\varepsilon)$ meets I. Consider $T_k := \bigcap_{\varepsilon\in\mathbb{Q}_+} G^k(\varepsilon) \subseteq I_k$; then, by Baire's Theorem, $I_k \backslash T_k$ is meagre. Put

$$T := T_0 \backslash \bigcup\nolimits_{k\in\mathbb{N}} (I_k \backslash T_k) : \qquad T \cap I_k \subseteq T_k \qquad (k \in \mathbb{N}).$$

As T is co-meagre, we may assume w.l.o.g. that the one-sided uniformity condition (1) holds on T.

Given $\varepsilon > 0$ and $t \in T$, by upper-semicontinuity of f at t, pick $r \in \mathbb{N}$ such that $t \in I_r$ and $f(u) < f(t) + \varepsilon$ for all $u \in I_r$. Now, as $t \in T_r$, $t \in G^r(\varepsilon)$, so we may pick $q \in \mathbb{D} \cap I_r$ and $\delta > 0$ with $t \in (q, q+\delta) \subseteq I_r$ and $m \in \mathbb{N}$ such that

$$f_n(s) < f(q) + \varepsilon \qquad (n > m,\ s \in I_\delta^+(q)),$$

again as $q \in S$. Now choose $d > 0$ such that $I_d(t) \subseteq (q, q+\delta)$. Then for $n > m$ and $s \in I_d(t)$

$$f_n(s) < f(q) + \varepsilon < f(t) + 2\varepsilon,$$

since $q \in I_r$. As $\varepsilon > 0$ was arbitrary,

$$f(t) = \lim_{\delta \downarrow 0} \limsup_n \sup\{f_n(s) : s \in I_\delta(t)\} \text{ for } t \in T. \qquad \square$$

Before proceeding further we need to extend the Beurling function η^φ some way to the left of the (natural) origin as follows (recalling from §1 the condition ($SE_{\mathbb{A}}$)); cf. BGT (2.11.2). Here we see the critical role of the *Popa* origin $\rho^* = -\rho^{-1}$ of §§1,3: the domain of the limit operation $\lim_{x\to\infty} \eta_x^\varphi(s)$, used to extend η^φ, is $\mathbb{G}_+^\rho$, i.e. s has to be to the right of the Popa origin.

Lemma 1 (Uniform Involutive Extension). *For $\varphi \in SE$, $\circ = \circ_\rho$ with $\rho = \rho_\varphi > 0$, put*

$$\eta^\varphi(t_\circ^{-1}) = \eta^\varphi(-t/\eta^\varphi(t)) := 1/\eta^\varphi(t), \qquad (t > 0);$$

then ($SE_{\mathbb{A}}$) holds for $\mathbb{A} = \mathbb{G}_+^\rho = (\rho^*, \infty)$. Moreover, this is a maximal positive extension: for each $s < \rho^*$, assuming $\varphi(x + s\varphi(x)) > 0$ is defined for all large x,

$$\lim_{x\to\infty} \eta_x^\varphi(s) = \lim_{x\to\infty} \varphi(x + s\varphi(x))/\varphi(x) = 0 = \eta(\rho^*).$$

Proof. Fixing $t > 0$ and taking $y := x + t\varphi(x)$ and $s_x = t/\eta_x(y)$ gives $x = y - s_x\varphi(y)$. Now

$$1/\eta_x(y) = \frac{\varphi(x)}{\varphi(x + t\varphi(x))} \to 1/\eta^\varphi(t) = \eta^\varphi(-t/\eta(t)).$$

So $s_x \to s = t/\eta(t)$ and

$$\frac{\varphi(y - s_x\varphi(y))}{\varphi(y)} = \frac{\varphi(x)}{\varphi(x + t\varphi(x))} \to 1/\eta^\varphi(t) = \eta^\varphi(-t/\eta^\varphi(t)) = \eta^\varphi(-s).$$

So for $s > 0$ with $\eta^{\varphi}(-s) > 0$ and y so large that $y(1 - s\varphi(y)/y) > 0$,

$$\frac{\varphi(y - s\varphi(y))}{\varphi(y)} \to \eta(-s) \text{ locally uniformly in } s \text{ for } \eta(-s) > 0.$$

As for the maximality assertion (even allowing $\mathbb{R}$ to be the domain of φ), since

$$0 \leqslant \liminf_{x\to\infty} \varphi(x + s\varphi(x))/\varphi(x) \leqslant \limsup_{x\to\infty} \varphi(x + s\varphi(x))/\varphi(x),$$

suppose there are $s < \rho^*$ and a divergent sequence x_n with

$$\tilde{\eta}(s) := \lim_{n\to\infty} \varphi(x_n + s\varphi(x_n))/\varphi(x_n) > 0,$$

including here the case $\tilde{\eta}(s) = +\infty$. As above, take $y_n = x_n + s\varphi(x_n)$ and $s_n = -s\varphi(x_n)/\varphi(x_n + s\varphi(x_n)) > 0$; then $x_n = y_n - s\varphi(x_n) = y_n + s_n\varphi(y_n)$ and $s_n \to -s\tilde{\eta}(s)^{-1} \geqslant 0$, strictly so unless $\tilde{\eta}(s) = +\infty$. So

$$\tilde{\eta}(s)^{-1} = \lim \varphi(y_n + s_n\varphi(y_n))/\varphi(y_n) = \eta(-s\tilde{\eta}(s)^{-1}) = 1 - \rho s\tilde{\eta}(s)^{-1}.$$

If $\tilde{\eta}(s) = +\infty$, this is already a contradiction. If $0 < \tilde{\eta}(s) < \infty$ cross-multiplying by $\tilde{\eta}(s)$, yields $\tilde{\eta}(s) = 1 + \rho s < 1 + \rho\rho^* = 0$, again a contradiction. □

Remark. For $s > 0$ and large enough y the expression $y - s\varphi(y)$ is positive provided $s < \liminf x/\varphi(x)$, that is for $s > \rho^*$. This corresponds to $\varphi(x) = O(x)$; if, however, as in BGT §2.11, $\varphi(x) = o(x)$, then $\rho_{\varphi} = 0$, so that $\rho^* = -\infty$, and so s may be arbitrary.

Definitions. Recalling (§1) that $\Delta_t^{\varphi} h(x) := h(x + t\varphi(x)) - h(x)$, and, taking limits here and below as $x \to \infty$ (rather than sequentially as $n \to \infty$), put for $\varphi \in SE$ and $\rho = \rho_{\varphi}$

$$\begin{aligned} \mathbb{A}^{\varphi} &:= \{t > \rho^* : \Delta_t^{\varphi} h \text{ converges to a finite limit}\}, \\ \mathbb{A}_{\mathrm{u}} &:= \{t > \rho^* : \Delta_t^{\varphi} h \text{ converges to a finite limit locally uniformly near } t\}. \end{aligned}$$

(For $\mathbb{A}^{\varphi} \subseteq \mathbb{G}_+^{\rho}$, see Lemma 1 above and Prop. 6 below.) So $0 \in \mathbb{A}^{\varphi}$, but we cannot yet assume either that $\mathbb{A}^{\varphi}$ is a subgroup, or that $0 \in \mathbb{A}_{\mathrm{u}}$, a critical point in Proposition 6 below. In the Karamata case $\varphi \equiv 1$, $\mathbb{A}^{\varphi} = \mathbb{A}^1$ is indeed a subgroup (see [BinO12, Prop. 1]).

For $t \in \mathbb{A}^{\varphi}$ put
$$K(t) := \lim_{x\to\infty} \Delta_t^{\varphi} h. \tag{K}$$
So $K(0) = 0$.

Proposition 5 below is included to help in reading the subsequent Proposition 6 – dedicated to checking when $\mathbb{A} \subseteq \mathbb{G}$ is a subgroup of a Popa group – which needs a sequential characterization of uniform convergence near a non-zero t (as $t_n \to t$ iff $c_n = t_n/t \to 1$); the proof is routine, so omitted.

Proposition 5. *$h(x+t\varphi(x))-h(x)$ converges locally (right-sidedly) uniformly to $K(t)$ near $t \neq 0$, iff for each divergent x_n and any $c_n \to 1$ ($c_n \downarrow 1$)*
$$h(x_n \circ_{\varphi} c_n t) - h(x_n) \to K(t);$$
then, taking suprema over sequences $c = \{c_n\} \downarrow 1$ and $x = \{x_n\} \to \infty$,
$$K(t) := \sup_{c,x}\{\limsup_{n\to\infty} h(x_n \circ_{\varphi} c_n t) - h(x_n)\}.$$

Proposition 6. *For $\varphi \in SE$, $\mathbb{A}_{\mathrm{u}}$ is a subgroup of $\mathbb{G}_+^{\rho}$ for $\rho = \rho_{\varphi}$ iff $0 \in \mathbb{A}_{\mathrm{u}}$; then $K : (\mathbb{A}_{\mathrm{u}}, \circ) \to (\mathbb{R}, +)$, defined by (K) above, is a homomorphism.*

Proof. We show that $v \circ_{\eta} u \in \mathbb{A}_{\mathrm{u}}$ for $u, v \in \mathbb{A}_{\mathrm{u}}$ with $v \circ_{\eta} u \neq 0$, and that $\mathbb{A}_{\mathrm{u}}$ is closed under inverses $u_{\circ}^{-1}$ for non-zero u, so it is a subgroup of $\mathbb{G}$ iff $1_{\mathbb{G}} = 0 \in \mathbb{A}_{\mathrm{u}}$. For $u, v \in \mathbb{A}_{\mathrm{u}}$, since $\eta_x(v) = \varphi(x + v\varphi(x))/\varphi(x) \to \eta(v)$,
$$u_v := u\eta(v)/\eta_x(v) \to u,$$
and so with $y = x \circ_{\varphi} v$, since by Prop. 2(iii) $x \circ_{\varphi} (v \circ_{\eta} u) = y \circ_{\varphi} u_v$,
$$\begin{aligned} h(x \circ_{\varphi} (v \circ_{\eta} u)) - h(x) &= [h(y \circ_{\varphi} u_v) - h(y)] + [h(x \circ_{\varphi} v) - h(x)] \\ &\to K(u) + K(v), \end{aligned}$$
i.e.
$$K(v \circ_{\eta} u) = \lim[h(x \circ_{\varphi} (v \circ_{\eta} u)) - h(x)] = K(u) + K(v).$$
As the convergence at u, v on the right occurs uniformly near u, v respectively, this is uniform near $v \circ u$, using Prop. 5 provided $v \circ u \neq 0$.

For non-zero $t \in \mathbb{A}_{\mathrm{u}}$, this time put $y := x \circ_{\varphi} t$; then, by Prop. 2(iv), $x = y \circ_{\varphi} t_{\varphi x}^{-1}$, so
$$h(y \circ_{\varphi} t_{\varphi x}^{-1}) - h(y) = [h(x) - h(y)] = -[h(x \circ_{\varphi} t) - h(x)] \to -K(t).$$

So, since $t_{\varphi x}^{-1} = -t/\eta_x(t) \to -t/\eta(t)$,

$$K(t_\circ^{-1}) = K(-t/\eta(t)) = \lim[h(y \circ_\varphi t_{\varphi x}^{-1}) - h(y)] = -K(t).$$

That is $t_\circ^{-1} \in \mathbb{A}_{\mathrm{u}}$ (and $K(t_\circ^{-1}) = -K(t)$); again this is locally uniform at $t \neq 0$, using Prop. 5. □

Theorem 3 (a corollary of Proposition 6) and Theorem 4 below, together with the results of §6 below, are of *dichotomy* type. The theme is that uniformity holds nowhere or (under assumptions) everywhere.

Theorem 3. *If $\mathbb{A}_{\mathrm{u}}$ is non-empty, then $0 \in \mathbb{A}_{\mathrm{u}}$ and so $\mathbb{A}_{\mathrm{u}}$ is a subgroup.*
In particular, for $h(t) = \log\varphi(t)$, if $\eta_x^\varphi(t) \to \eta(t)$ locally uniformly near t for some $t > 0$, then this convergence is locally uniform near t for all $t \geqslant 0$.

Proof. Choose $s \in \mathbb{A}_{\mathrm{u}}$, which without loss of generality is non-zero (otherwise there is nothing to prove). So, as above, $t := -s/\eta(s) \in \mathbb{A}_{\mathrm{u}}$. For arbitrary $z_n \to 0$ and x_n divergent, take $s_n := s + z_n$ and $t_n := -s/\eta_{x(n)}(s_n) \to t$; then $y_n = x_n + (s + z_n)\varphi(x_n)$ is divergent. So (since $s\varphi(x_n) = (s/\eta_{x(n)}(s_n))\varphi(y_n)$)

$$h(x_n + z_n\varphi(x_n)) - h(x_n) = h(x_n + (s + z_n)\varphi(x_n) - (s/\eta_{x(n)}(s_n))\varphi(y_n)) - h(x_n),$$

which (as $y_n = x_n + (s + z_n)\varphi(x_n)$) is

$$\begin{aligned}
&= h(y_n - s/\eta_{x(n)}(s_n)\varphi(y_n)) - h(y_n) + h(x_n + s_n\varphi(x_n)) - h(x_n) \\
&= h(y_n \circ_\varphi t_n) - h(y_n) + h(x_n \circ_\varphi s_n) \\
&\to h(t) + h(s) = h(s_\varphi^{-1}) + h(s) = 0.
\end{aligned}$$

So $\Delta_t^\varphi h$ converges locally near 0, i.e. $0 \in \mathbb{A}_{\mathrm{u}}$ – a subgroup, by Prop. 6.

In particular, for $h = \log\varphi$,

$$h(x_n + z_n\varphi(x_n)) - h(x_n) \to 0 \text{ iff } \varphi(x_n + z_n\varphi(x_n))/\varphi(x_n) \to 1,$$

and $\mathbb{A}_{\mathrm{u}}$ is non-empty as $\varphi \in SE$. □

The following result extends the Uniformity Lemma of [BinO10, Lemma 3]. Although the proof parallels the original, the current one-sided context demands the closer scrutiny offered here. To describe more accurately the convergence in (K) above, we write

$$\begin{aligned}
\Delta_t^\varphi h(x) &\to K_+(t) \text{ if uniform near } t \text{ on the right,} && (K_+) \\
\Delta_t^\varphi h(x) &\to K_-(t) \text{ if uniform near } t \text{ on the left,} && (K_-) \\
\Delta_t^\varphi h(x) &\to K_\pm(t) \text{ if uniform near } t. && (K_\pm)
\end{aligned}$$

Lemma 2. (i) *For* $\varphi \in SE$:

(a) *if the convergence in* (K) *is uniform (resp. right-sidedly uniform) near* $t=0$*, then it is uniform (resp. right-sidedly uniform) everywhere in* $\mathbb{A}^{\varphi}$ *and for* $u \in \mathbb{A}^{\varphi} \cap (0,\infty)$

$$K_{+}(u) = K(u) + K_{+}(0);$$

(b) *if the convergence in* (K) *is uniform near* $t = u \in \mathbb{A}^{\varphi} \cap (\mathbb{A}^{\varphi})_{\circ}^{-1} \cap (0,\infty)$, *then it is uniform near* $t=0$:

$$K_{\pm}(0) = K_{\pm}(u) + K(u_{\circ}^{-1});$$

(ii) *if* $\rho = 0$ *and* $\varphi \in SN$ *is monotonic increasing, and the convergence in* (K) *is right-sidedly uniform near* $t = u \in \mathbb{A}^{\varphi} \cap (0,\infty)$, *then it is right-sidedly uniform near* $t=0$:

$$K_{+}(0) = K_{+}(u) + K(u_{\circ}^{-1}).$$

Proof. (i) (a) Suppose (K) holds locally right-sidedly uniformly (uniformly) near $t=0$. Let $u \in \mathbb{A}^{\varphi}$ and $z_n \downarrow 0$ (resp. $z_n \to 0$). For x_n divergent ($x_n \to \infty$), $y_n := x_n \circ_{\varphi} u = x_n(1 + u\varphi(x_n)/x_n)$ is divergent and

$$h(x_n \circ_{\varphi} (u+z_n)) - h(x_n) = h(x_n \circ_{\varphi} u) - h(x_n) + h(y_n \circ_{\varphi} z_n/\eta_{x(n)}(u)) - h(y_n). \quad (*)$$

Without loss of generality $\eta_{x(n)}(u) > 0$ (all n), since $u \in \mathbb{G}_{+}^{\rho}$ and so

$$\eta_{x(n)}(u) \to \eta(u) > 0;$$

then $z_n/\eta_{x(n)}(u) \downarrow 0$ (resp. $z_n/\eta_{x(n)}(u) \to 0$). From $h(x_n \circ_{\varphi} u) - h(x_n) \to K(u)$, and the assumed uniform behaviour at the origin, there is right-sidedly uniform (uniform) behaviour near u. The second statement follows on specializing to $u \in \mathbb{A}^{\varphi} \cap (0,\infty)$ and taking limits in $(*)$.

(b) For the converse we argue as in Theorem 3. Suppose uniformity holds near $u \in \mathbb{A}^{\varphi} \cap (\mathbb{A}^{\varphi})^{-1} \cap (0,\infty)$; then $v := u_{\circ}^{-1} = -u/\eta(u) \in \mathbb{A}^{\varphi} \cap (\rho^{*}, 0)$. Let $z_n \to 0$; then $z_n' := z_n/\eta_{x(n)}(v) \to 0$, as $\eta_{x(n)}(v) \to \eta(v)$. Also $(-v)/\eta_{x(n)}(v) \to (-v)/\eta(v) = v_{\circ}^{-1} = u$, so

$$\lim (-v + z_n)/\eta_{x(n)}(v) = u + 0.$$

Taking $y_n := x_n \circ_{\varphi} v$ ($< x_n$ for $v < 0$,as here)

$$x_n \circ_{\varphi} z_n = (x_n \circ_{\varphi} v) \circ_{\varphi} (-v + z_n)/\eta_{x(n)}(v),$$

and

$$\begin{aligned} h(x_n \circ_\varphi z_n) - h(x_n) &= h(y_n \circ_\varphi (-v + z_n)/\eta_{x(n)}(v)) - h(y_n) + h(x_n \circ_\varphi v) - h(x_n) \\ &\to K(u) + K(v) = K(u) + K(u_\circ^{-1}), \end{aligned}$$

where the convergence on the right is uniform in the first term and pointwise in the second term.

(ii) When $\varphi \in SN$ is monotone, the argument in (b) above may be amended to deal with right-sided convergence, as $1/\eta_{x(n)}(v) = \varphi(x_n)/\varphi(y_n) \geqslant 1$ (for $v < 0$), and so $1/\eta_{x(n)}(v)$ tends to 1 from above, as $\rho = 0$. Also $z_n' = z_n$, so if $z_n \downarrow 0$, then $z_n\varphi(x_n)/\varphi(y_n)$ tends to 0 from above, since $z_n \geqslant 0$ and

$$(-v + z_n)/\eta_{x(n)}(v) \text{ tends to } u \text{ from above,}$$

as $(-v)/\eta_{x(n)}(v)$ tends from above to $(-v) = u > 0$. From here the argument is valid when 'uniform' is replaced by 'right-sidedly uniform'. $\square$

Remark. For $\varphi \in SE$ and $\eta = \eta^\varphi$ write $\varphi \in SE^+/SE^-$ (for $u > 0$) respectively according as

$$\varphi(x + u\varphi(x))/\varphi(x) \text{ tends to } \eta(u) \text{ from below, or from above}$$

as $x \to \infty$, and likewise for $\varphi \in SN$ (with $\eta^\varphi \equiv 1$) and SN^-. So if $\varphi \in SN$ and φ is increasing, then $\varphi \in SN^-$, since $\varphi(x + u\varphi(x)) > \varphi(x)$ for $u > 0$, so

$$\varphi(x + u\varphi(x))/\varphi(x) \text{ tends to 1 from above.}$$

This was used in (ii) above, and extends to SE. Of course $\eta \in SE^+ \cap SE^-$.

The next result leads from a one-sided condition to a two-sided conclusion. This is the prototype of further such results, useful later.

Theorem 4. *If the pointwise convergence* (K) *holds on a co-meagre set in* $\mathbb{G}_+^\rho$ *with the limit function* K *upper semicontinuous also on a co-meagre set, and the one-sided condition*

$$K(t) = \lim_{\delta \downarrow 0} \limsup_{x\to\infty} \sup\{h(x + s\varphi(x)) - h(x) : s \in I_\delta^+(t)\} \qquad (UNIF^+)$$

holds at the origin – then two-sided limsup convergence holds everywhere:

$$\mathbb{A}^\varphi = \mathbb{A}_{\mathrm{u}} = \mathbb{G}_+^\rho.$$

Proof. The pointwise convergence assumption says $\mathbb{A}^{\varphi}$ is co-meagre (in $\mathbb{G}_+^{\rho}$); w.l.o.g. $\mathbb{A}^{\varphi}=(\mathbb{A}^{\varphi})_{\circ}^{-1}$, otherwise work below with the co-meagre set $\mathbb{A}^{\varphi}\cap(\mathbb{A}^{\varphi})_{\circ}^{-1}$. Take $f(t):=K(t)$; then $f_n(t):=h(x_n\circ_{\varphi}t)-h(x_n)\to f(t)$ holds pointwise quasi everywhere on $\mathbb{A}^{\varphi}$. Since $(UNIF^{+})$ holds at $t=0$, by Lemma 2(i)(a), it holds everywhere in $\mathbb{A}^{\varphi}$ and so quasi everywhere. By Proposition 4, its two-sided limsup version holds quasi everywhere, and so at some point $u\in\mathbb{A}^{\varphi}\cap(\mathbb{A}^{\varphi})_{\circ}^{-1}\cap(0,\infty)$. Then by Lemma 2(i)(b) the two-sided limsup version holds at 0, and so by Lemma 2(i)(a) it holds everywhere in $\mathbb{A}^{\varphi}$. It now follows that $0\in\mathbb{A}^{\varphi}=\mathbb{A}_{\mathrm{u}}$ and so $\mathbb{A}_{\mathrm{u}}$ is a co-meagre subgroup of $\mathbb{G}_+^{\rho}$; so, by the Steinhaus Subgroup Theorem (see [BinO9]), which applies here by Prop. 6, $\mathbb{A}_{\mathrm{u}}=\mathbb{G}_+^{\rho}$. □

6 Dichotomy

We continue with the setting of §5, but here we assume less about $\mathbb{A}^{\varphi}$ – in place of being co-meagre we ask that it contains a non-meagre Baire subset $S\subseteq\mathbb{G}_+^{\rho}$. This is a local version of the situation in §5 in that
(i) S is locally co-meagre quasi everywhere, and
(ii) $\mathbb{A}^{\varphi}$ is non-meagre and contains a Baire subset to witness this.
For general h and φ we cannot assume this happens. However, under certain axioms of set-theory this will be guaranteed: see §11. Now $\langle S\rangle$, the $\circ$-subgroup generated by S, will of course be $\mathbb{G}_+^{\rho}$, again by the Steinhaus Subgroup Theorem, as in Theorem 4. So our aim here is to verify that $\mathbb{A}^{\varphi}$ is a subgroup by checking that $\mathbb{G}_+^{\rho}=\langle S\rangle\subseteq\mathbb{A}_{\mathrm{u}}\subseteq\mathbb{A}^{\varphi}$.

Theorem 5. *For $\varphi\in SE$ and h Baire, if $\mathbb{A}^{\varphi}$ contains a non-meagre Baire subset, then $\mathbb{A}^{\varphi}=\mathbb{G}_+$ and K is a homomorphism: $K(u)=c\log(1+\rho t)$, for some $c\in\mathbb{R}$, $(u\in\mathbb{G}_+)$, if $\rho=\rho_{\varphi}>0$.*

Given our opening remarks, this reads as an extension of the Fréchet-Banach Theorem on the continuity of Baire/measurable additive functions – for background see [BinO9]. The proof (see below) parallels Prop. 1 of [BinO12], extending the cited result from the Karamata to the Beurling setting, but now we need the Baire property to employ uniformity arguments here.

Proposition 7 extends Theorem 7 (UCT) of [BinO10] and is crucial here.

Proposition 7 (Uniformity). *Suppose $S\subseteq\mathbb{A}^{\varphi}$ for some Baire non-meagre S. Then for Baire h the convergence in (K) of §5 is uniform near $u=0$ and* *so also near $u=t$ for $t\in S$, i.e. $S\subseteq \mathbb{A}_{\mathrm{u}}\subseteq \mathbb{A}^{\varphi}$.*

Proof . For each n, define for $t>\rho^*$ the function $k_n(t):=h(n\circ_{\varphi} t)-h(n)$, which is Baire; then for $t\in\mathbb{A}^{\varphi}$

$$K(t)=\lim\nolimits_n k_n(t),$$

and so $k=K|S$ is a Baire function with non-meagre domain. Now apply the argument of Theorem 7 of [BinO10] to S and k as defined here (so that Baire's Continuity Theorem [Oxt, Th. 8.1] applies to the Baire function k), giving uniform convergence near $u=0$, so uniform convergence near any $u\in S$, by Lemma 2(i)(a). $\square$

Corollary 4. *If $S\subseteq\mathbb{A}^{\varphi}$ with $S\subseteq\mathbb{G}_+^{\rho}$ Baire and non-meagre, and $\rho\geqslant 0$, then*
(i) $S_{\circ}^{-1}=\{-s/(1+\rho s):s\in S\}\subseteq\mathbb{A}^{\varphi}$*;*
(ii) $S\circ S=\{s+t\eta(s):s,t\in S\}\subseteq\mathbb{A}^{\varphi}$.

Proof. (i) As $S_{\circ}^{-1}$ is Baire and non-meagre, Prop. 7 applies and $S_{\circ}^{-1}\subseteq\mathbb{A}_{\mathrm{u}}\subseteq\mathbb{A}^{\varphi}$.
(ii) By Th. PJ, $S\circ S$ is isomorphic either to $S+S$ (for $\rho=0$) or to $\eta_{\rho}(S)\eta_{\rho}(S)$ (for $\rho>0$) and so is Baire and non-meagre, by the Steinhaus Sum Theorem ([BinO9]); again Prop. 7 applies and $S\circ S\subseteq\mathbb{A}_{\mathrm{u}}\subseteq\mathbb{A}^{\varphi}$. $\square$

Proof of Theorem 5. Suffice it to assume $\rho=\rho_{\varphi}>0$. Replacing S by $S\cup(S_{\circ}^{-1})$ if necessary, we may assume by Cor. 4 that S is symmetric ($S=S_{\circ}^{-1}$), and w.l.o.g. $0=1_{\mathbb{G}}\in S$, by Prop. 7.

Applying Cor. 4(ii) inductively, we deduce that

$$S^*:=\bigcup\nolimits_{n\in\mathbb{N}}(n)\circ S\subseteq\mathbb{A}^{\varphi},$$

where $(n)\circ S$ denotes $S\circ_{\eta}...\circ_{\eta}S$ to n terms. So S^* is symmetric, and a semigroup: if $s\in(n)\circ S$ and $s'\in(m)\circ S$, then $s\circ s'\in(n+m)\circ S\subseteq S^*$. So $\mathbb{A}^{\varphi}$ contains S^*. As $0\in S^*$ (as above), S^* is a subgroup (being symmetric, since $\circ$ is commutative); hence S^* is all of $\mathbb{G}_+^{\rho}$. So $S^*=\mathbb{G}_+^{\rho}=\mathbb{A}_{\mathrm{u}}=\mathbb{A}^{\varphi}$. By Prop. 6, $\bar{K}(t)=K(\eta^{-1}(e^t))$ is additive on $\mathbb{R}$; indeed, by Prop. 6 with $\eta(u)=e^x$ and $\eta(v)=e^y$

$$\bar{K}(x+y):=K(\eta^{-1}(e^{x+y}))=K(u\circ_{\eta}v)=K(u)+K(v)=\bar{K}(x)+\bar{K}(y).$$

By Prop. 7 convergence is uniform near $u = 0$, so that $\bar{K}(t)$ is bounded in a neighbourhood of 0, and, being additive, is linear; see e.g. BGT 1.3, [Kuc], [BinO9,11]. So for some $c \in \mathbb{R}$:

$$c \log \eta(u) = c \log(1 + \rho u) = cx = \bar{K}(x) = \bar{K}(\eta^{-1}(e^x)) = K(u) \quad (u > \rho^*). \quad \square$$

7 Quantifier weakening

Here we again drop the assumption that $\mathbb{A}^\varphi$ is co-meagre; instead we will impose a density assumption, and employ a subadditivity argument developed in [BinO12]. To motivate this, we recall the following decomposition theorem of a function, with a one-sided finiteness condition, into two parts, one decreasing, one with suitable limiting behaviour.

Theorem BG2 ([BinG2, Th. 7]). *The following are equivalent:*
(i) *The function U has the decomposition*

$$U(x) = V(x) + W(x),$$

where V has linear limiting moving average K_V as in §1, and $W(x)$ is non-increasing;
(ii) *the following limit is finite:*

$$\lim_{\delta \downarrow 0} \limsup_{x \to \infty} \sup \left\{ \frac{U(x \circ_\varphi t) - U(x)}{\delta \varphi(x)} : t \in I_\delta^+(0) \right\} < \infty.$$

Definitions. For $\varphi \in SE$ and $\rho = \rho_\varphi$, put

$$\begin{aligned} H^\dagger(t) &:= \lim_{\delta \downarrow 0} \limsup_{x \to \infty} \sup \left\{ h(x \circ_\varphi s) - h(x) : s \in I_\delta^+(t) \right\} \qquad (t > \rho^*), \\ \mathbb{A}_{\mathrm{u}}^\dagger &:= \{ t > \rho^* : H^\dagger(t) < \infty \}. \end{aligned}$$

So $\mathbb{A}_{\mathrm{u}} \subseteq \mathbb{A}_{\mathrm{u}}^\dagger$, as $H^\dagger(t) = K(t)$ on $\mathbb{A}_{\mathrm{u}}$. In Theorem 6 below we apply the techniques of [BinO11,12]; a first step for this is the following. Here it is again convenient to rely on Prop. 5.

Proposition 8. *For $\varphi \in SE$ and $\eta = \eta^\varphi$, $H^\dagger$ is subadditive on $\mathbb{A}_{\mathrm{u}}^\dagger$ over non-inverse pairs of elements s, t:*

$$H^\dagger(s \circ_\eta t) \leqslant H^\dagger(s) + H^\dagger(t) \quad (s, t \in \mathbb{A}_{\mathrm{u}}^\dagger, s \circ_\eta t \neq 0).$$

If ρ^*_φ is an accumulation point of $\mathbb{A}^\dagger_{\mathrm{u}}$, then either $\liminf_{s\downarrow\rho^*} H^\dagger(s)$ *is infinite, or $H^\dagger \geqslant 0$ on $\mathbb{A}^\dagger_{\mathrm{u}}$.*

Proof. For $c = \{c_n\} \to 1$ and $x = \{x_n\}$ divergent, put

$$H(t; x, c) := \limsup h(x_n \circ_\varphi c_n t) - h(x_n) \quad (t \neq 0).$$

As in Prop. 6, for a given $c_n \to 1$ and divergent x_n, take $y_n := x_n \circ_\varphi c_n s$, $d_n := c_n\eta(s)\varphi(x_n)/\varphi(y_n) \to \eta(s)\eta(s)^{-1} = 1$. Now

$$x_n \circ_\varphi c_n(s + t\eta(s)) = x_n + c_n(s + t\eta(s))\varphi(x_n) = y_n + d_n t\varphi(y_n),$$

so

$$h(x_n \circ_\varphi c_n(s + t\eta(s))) - h(x_n) = h(y_n \circ_\varphi d_n t) - h(y_n) + h(x_n \circ_\varphi c_n s) - h(x_n),$$

whence

$$H(s + t\eta(s); c, x) \leqslant H(t; d, y) + H(s; c, x).$$

So $H(s \circ_\eta t; c, x) \leqslant H^\dagger(t) + H^\dagger(s)$, as $H(t; d, y) \leqslant H^\dagger(t)$ and $H(s; c, x) \leqslant H^\dagger(s)$. Now we may take suprema, since Prop. 5 applies provided $s \circ_\eta t \neq 0$.

For the final assertion, let $t \in \mathbb{A}^\dagger_{\mathrm{u}}$ and assume that $\liminf_{s\downarrow\rho^*} H^\dagger(s)$ is finite. Let $\varepsilon > 0$. Since $s \circ_\eta t \downarrow \rho^*$, for small enough s with $\rho^* < s < t_\circ^{-1}$

$$-H^\dagger(t) + \liminf_{s\downarrow\rho^*} H^\dagger(s) - \varepsilon \leqslant H^\dagger(s \circ_\eta t) - H^\dagger(t) \leqslant H^\dagger(s).$$

So

$$-H^\dagger(t) + \liminf_{s\downarrow\rho^*} H^\dagger(s) - \varepsilon \leqslant \liminf_{s\downarrow\rho^*} H^\dagger(s) : \quad -\varepsilon \leqslant H^\dagger(t).$$

So $0 \leqslant H^\dagger(t)$, as $\varepsilon > 0$ was arbitrary. □

Our next result clarifies the role of the Heiberg-Seneta condition, for which see BGT §3.2.1 and [BinO12]. It is here that we again use $(SE_{\mathbb{A}})$ on the set $\mathbb{A} = \mathbb{G}^\rho_+ = \{t : \eta_\rho(t) > 0\}$ with $\rho = \rho_\varphi$.

Proposition 9. *For $\varphi \in SE$, the following are equivalent:*
(i) $0 \in \mathbb{A}_{\mathrm{u}}$ *(i.e. $\mathbb{A}_{\mathrm{u}} \neq \emptyset$ and so a subgroup);*
(ii) $\lim_{x\to\infty}[h(x + u\varphi(x)) - h(x)] = 0$ *uniformly near $u = 0$;*
(iii) $H^\dagger(t)$ *satisfies the two-sided Heiberg-Seneta condition:*

$$\limsup_{u\to 0} H^\dagger(u) \leqslant 0. \qquad (HS_\pm(H^\dagger))$$

Proof. It is immediate that (i) and (ii) are equivalent. We will show that (ii) and (iii) are equivalent. First assume the Heiberg-Seneta condition. Take $\varepsilon > 0$, x_n divergent, and z_n null (i.e. $z_n \to 0$). By $HS_\pm(H^\dagger)$, there is $\delta_\varepsilon > 0$ such that

$$H^\dagger(t) < \varepsilon \quad (0 < |t| < \delta_\varepsilon).$$

So for each t with $0 < |t| < \delta_\varepsilon$ there are $\delta(t) > 0$ and X_t such that

$$h(x \circ_\varphi s) - h(x) < \varepsilon \qquad (x > X_t, s \in I^+_{\delta(t)}(\pm t)).$$

By compactness, there are: $\delta > 0$, a finite set F of points t with $\delta_\varepsilon/3 \leqslant t \leqslant 2\delta_\varepsilon/3$, and X such that

$$h(x \circ_\varphi s) - h(x) < \varepsilon \qquad (x > X, s \in I^+_\delta(\pm t), t \in F),$$

and, further, $\{I^+_\delta(\pm t) : t \in F\}$ covers $[-2\delta_\varepsilon/3, -\delta_\varepsilon/3] \cup [\delta_\varepsilon/3, 2\delta_\varepsilon/3]$. By assumption, $\eta_{x(n)}(s) \to 1$ uniformly as $s \to 0$, so we may fix $t, t' \in F$ and $s > 0$ such that:
(i) $s \in (t, t+\delta)$,
(ii) w.l.o.g., for all n, $s_n = s + z_n \in I^+_\delta(t)$,
(iii) w.l.o.g., for all n, $-s/\eta_{x(n)}(s_n) \in I^+_\delta(-t')$.
Take $y_n = x_n \circ_\varphi s_n$; then for $x_n, y_n > X$, as in the proof of Theorem 3,

$$\begin{aligned}
& h(x_n + z_n\varphi(x_n)) - h(x_n) \\
= \; & h(x_n + s_n\varphi(x_n) - s\varphi(x_n)) - h(x_n) \\
= \; & h(y_n - s/\eta_{x(n)}(s_n)\varphi(y_n)) - h(y_n) + h(x_n \circ_\varphi s_n) - h(x_n) \leqslant 2\varepsilon.
\end{aligned}$$

In summary: for any divergent x_n and null z_n

$$h(x_n + z_n\varphi(x_n)) - h(x_n) < 2\varepsilon \qquad \text{for all large } n. \tag{2}$$

Towards a similar lower bound, suppose that for some divergent y_n and null z'_n

$$h(y_n \circ_\varphi z'_n) - h(y_n) \leqslant -2\varepsilon \qquad \text{for all } n.$$

Take $x_n := y_n \circ_\varphi z'_n$, which is divergent; then $y_n = x_n \circ_\varphi z_n$ for $z_n := -z'_n\varphi(y_n)/\varphi(x_n)$ which is null, since $\varphi(y_n)/\varphi(y_n \circ_\varphi z'_n) \to 1$ (by locally uniform convergence of $\eta_{y(n)}$ near 0). So for all n

$$h(x_n) - h(x_n + z_n\varphi(x_n)) \leqslant -2\varepsilon : \qquad h(x_n + z_n\varphi(x_n)) - h(x_n) \geqslant 2\varepsilon,$$

a contradiction to (2) for n large enough.

So the Heiberg-Seneta condition yields

$$\lim[h(x+u\varphi(x))-h(x)]=0 \text{ uniformly near } u=0,$$

i.e. (ii) holds.

Conversely, assuming (ii), for given $\varepsilon>0$ there are $X>0$ and $d>0$ so that for $x>X$ and $|u|<d, h(x+u\varphi(x))-h(x)<\varepsilon$. So for $x>X$

$$\sup\{h(x+u\varphi(x))-h(x):|u|<d\}\leqslant\varepsilon.$$

Fixing $t\in(-d,d)$, choose $\delta>0$ so small that $I_\delta(t)\subseteq(-d,d)$; then

$$H_\delta(t):=\limsup_{x\to\infty}\sup\{h(x+u\varphi(x))-h(x):u\in I_\delta^+(t)\}\leqslant\varepsilon.$$

But $H_\delta(t)$ is decreasing with δ; so $H^\dagger(t)=\lim_{\delta\downarrow 0}H_\delta(t)\leqslant\varepsilon$ for $t\in(-d,d)$, i.e. $\limsup_{u\to 0}H^\dagger(u)\leqslant 0$. □

The final result of this section is the Beurling version of a theorem proved in the Karamata framework of [BinO12]. However, uniformity plays no role there, whereas here it is critical. The result shows that weakening the quantifier in the definition of additivity to range only over a dense subgroup, determined by locally uniform limits, yields 'linearity' of $H^\dagger$. The K in Th. 6 below is as in (K) of §5, cf. Prop. 6.

Theorem 6 (Quantifier Weakening from Uniformity). *For $\rho>0$, if $\mathbb{A}_{\mathrm{u}}$ is dense in $\mathbb{G}_+^\rho$ and $H^\dagger(t)=K(t)$ on $\mathbb{A}_{\mathrm{u}}$ – i.e. $H^\dagger:(\mathbb{A}_{\mathrm{u}},\circ_\rho)\to(\mathbb{R},+)$ is a homomorphism – then $\mathbb{A}_{\mathrm{u}}=\mathbb{G}_+^\rho$ and for some $c\in\mathbb{R}$:*

$$H^\dagger(t)=c\log(1+\rho t)\quad(t>\rho^*).$$

Proof. We check that Theorem 1 of [BinO12] applies respectively to $\bar{H}(t):=H^\dagger(\eta^{-1}(e^t))$ and $\bar{K}(t):=K(\eta^{-1}(e^t))$ in place of H and K there, and with $\mathbb{A}:=\eta^{-1}(\exp[\mathbb{A}_{\mathrm{u}}])$, which is dense in $\mathbb{R}$, since η is an isomorphism taking $(\mathbb{G}_+^\rho,\circ_\rho)$ to $(\mathbb{R}_+,\cdot)$ (by Theorem PJ). Indeed, as in Theorem 5 $\bar{K}$ is additive on $\mathbb{R}$ (by Prop. 6), and likewise, by Prop. 8, $\bar{H}$ is subadditive. As $e^0=1=\eta(0)$, $\bar{H}$ satisfies the Heiberg-Seneta condition, by Prop. 9. Finally, since $H^\dagger(t)=K(t)$ on $\mathbb{A}_{\mathrm{u}}$, $\bar{H}(t)=\bar{K}(t)$ on $\mathbb{A}$. So $\bar{K}$ is linear by [BinO12, Th. 1], and the conclusion follows once again as in Theorem 5. □

Remark. As $\log[1+\rho(u\circ_\rho v)]=\log[(1+\rho u)(1+\rho v)]$, the function $c\log(1+\rho t)$ is 'subadditive' in the sense of Prop. 8 (indeed, perhaps 'additive').

8 Representation

We begin by identifying the limiting moving average K_F of §1. Below φ, being increasing, is Baire.

Lemma 3. *If $\varphi \in SE$ is increasing and the following limit exists for $F : \mathbb{R} \to \mathbb{R}$:*

$$K_F(u) := \lim \frac{F(x \circ_\varphi u) - F(x)}{\varphi(x)}, \qquad (u > \rho_\varphi^*)$$

– then K_F as above satisfies

$$K_F(u \circ_\eta v) = K_F(u) + K_F(v)\eta(u) \text{ for } \eta = \eta^\varphi;$$

if F is Baire/measurable, then K_F and $\eta = \eta^\varphi$ are of the form

$$K_F(u) = c_F u, \;\; \eta(u) = 1 + \rho u.$$

Proof. Write $y = x + u\varphi(x)$; then $\varphi(y)/\varphi(x) \to \eta(u)$. Now

$$\begin{aligned} \frac{F(x \circ_\varphi [u+v]) - F(x)}{\varphi(x)} &= \frac{F(y \circ_\varphi [v\varphi(x)/\varphi(y)]) - F(y)}{\varphi(y)} \frac{\varphi(y)}{\varphi(x)} \\ &\quad + \frac{F(x \circ_\varphi u) - F(x)}{\varphi(x)}. \end{aligned}$$

Write $w := v/\eta(u)$; then, taking limits above, gives

$$K_F(u + w\eta(u)) = K_F(w)\eta(u) + K_F(u).$$

Assuming F is Baire/measurable, $K_F(t) = \lim_{n\to\infty}[(F(n \circ_\varphi u) - F(x))/\varphi(n)]$ is Baire/measurable (as in Prop. 7). By [BinO11, Th.9,10] $K_F(x) = c_F H_0(x)$, where $H_0(x) := x$. So $K_F(u) = c_F u$, for some c_F.□

The result above formally extends to (i) the Beurling framework, and (ii) to the class SE the notion of Π_g-class, due to Bojanić-Karamata/de Haan, for which see BGT Ch. 3, since just as there

$$\text{(i)} \; \frac{F(x \circ_\varphi u) - F(x)}{\varphi(x)} \sim c_F H_0(u) : \qquad \text{(ii)} \; \frac{F(x \circ_\varphi u) - F(x)}{u\varphi(x)} \to c_F. \quad (\Pi_\varphi)$$

Definition. Say that F is of *Beurling Π_φ-class with φ-index* $c = c_F$ (cf. BGT Ch. 3) if the convergence in (Π_φ(ii)) is locally uniform in u.

This should be compared with Theorem BG2 in §7. We now use a Goldie-type argument (see [BinO11]) to establish the representation below for the class Π_φ.

Theorem 7 (Representation for Beurling Π_φ-class with φ-index c). *For F Baire/measurable, F is of additive Beurling Π_φ-class with φ-index c and $\varphi \in SE$ iff*

$$F(x) = b + cx + \int_1^x e(t)dt, \ b \in \mathbb{R} \text{ and } e \to 0.$$

Proof. As above, by the λ-UCT of [Ost3, Th. 1], there exists X such that for all $x \geqslant X$ and all u with $|u| \leqslant 1$

$$\frac{F(x \circ_\varphi u) - F(x)}{u\varphi(x)} = c + \varepsilon(x;u),$$

with

$$\varepsilon(x;u) \to 0 \text{ uniformly for } |u| \leqslant 1 \text{ as } x \to \infty.$$

Put

$$e(x) := \sup\{\varepsilon(x,u) : |u| \leqslant 1\};$$

then $e(x) \to 0$ as $x \to \infty$.

Using a Beck sequence $x_{n+1} = x_n \circ_\varphi u$ ([BinO11, §3]; cf. Bloom [Blo], BGT Lemma 2.11.2) starting at $x_0 = X$ and ending at $x_m = x(u) \leqslant x$ with $x < x(u) \circ_\varphi u = x_{m+1}$ yields

$$\begin{aligned}
F(x(u)) - F(X) &= \sum\nolimits_{n=0}^{m-1} F(x_{n+1}) - F(x_n) = \sum\nolimits_{n=0}^{m-1} (c + \varepsilon(x_n;u))u\varphi(x_n) \\
&= \sum\nolimits_{n=0}^{m-1} (c + \varepsilon(x_n;u))(x_n + u\varphi(x_n) - x_n) \\
&= c\sum\nolimits_{n=0}^{m-1} (x_{n+1} - x_n) + \sum\nolimits_{n=0}^{m-1} \varepsilon(x_n;u)(x_{n+1} - x_n) \\
&= c(x(u) - X) + \sum\nolimits_{n=0}^{m-1} \varepsilon(x_n;u)(x_{n+1} - x_n).
\end{aligned}$$

Since F is Baire/measurable, we may restrict attention to points x where F is continuous. For x fixed, note that $u\varphi(x_n) \leqslant u\varphi(x) \to 0$ as $u \to 0$, so $x(u) \to x$; taking limsup as $u \to 0$,

$$F(x) = F(X) + c(x - X) + \int_X^x e(t)dt,$$

with $e(x) \to 0$, as above. So on differencing,

$$\frac{F(x + u\varphi(x)) - F(x)}{u\varphi(x)} = c + \frac{1}{u\varphi(x)} \int_x^{x+u\varphi(x)} e(t)dt \to c.$$

So F is Beurling Π_φ-class with φ-index c iff it has the representation stated. $\square$

We note also a generalization of Prop. 8 and Lemma 2, for which we need notation (similar to that in §7) analogous to the Karamata Ω of BGT §3.0 (cf. BGT Th. 3.3.2/3).

Definitions. For $\varphi \in SE$ and $\rho = \rho_\varphi$, put

$$\begin{aligned} \Omega_h^\dagger(t) &:= \lim_{\delta\downarrow 0}\limsup_{x\to\infty}\sup\left\{(h(x\circ_\varphi s)-h(x))/\varphi(x) : s\in I_\delta^+(t)\right\}, \\ \mathbb{A}_\Omega^\dagger &:= \{t>\rho^* : \Omega_h^\dagger(t)<\infty\}. \end{aligned}$$

Proposition 8′. *For $\varphi \in SE$ and $\eta = \eta^\varphi$, $\Omega_h^\dagger$ is η-subadditive on $\mathbb{A}_\Omega^\dagger$:*

$$\Omega_h^\dagger(s\circ_\eta t) \leqslant \Omega_h^\dagger(t)\eta(s) + \Omega_h^\dagger(s) \quad (s,t\in\mathbb{A}_\Omega^\dagger, s\in(\rho_\varphi^*,+\infty)).$$

Proof. For $c=\{c_n\}\to 1$ and $x=\{x_n\}$ divergent, put

$$\Omega_h^\dagger(t;x,c) := \limsup[h(x_n\circ_\varphi c_n t)-h(x_n)]/\varphi(x_n).$$

As in Prop. 6, for a given $c_n\to 1$ and divergent x_n, take

$$y_n = x_n\circ_\varphi c_n s,\ d_n := 1/\eta_{x(n)}(s)\to\eta(s)^{-1}>0.$$

Since by Prop. 2(iii)

$$\begin{aligned} & [h(x_n\circ_\varphi c_n(s\circ_\eta t))-h(x_n)]/\varphi(x_n) \\ = \ & [h(y_n\circ_\varphi d_n t\eta(s))-h(y_n)]/\varphi(y_n)\cdot\eta_{x(n)}(c_n s)+[h(x_n\circ_\varphi c_n s)-h(x_n)]/\varphi(x_n), \end{aligned}$$

$$\Omega_h^\dagger(s+t\eta(s);c,x) \leqslant \Omega_h^\dagger(t;d,y)\eta(s)+\Omega_h^\dagger(s;c,y).$$

Now take suprema. $\square$

We note an extension of [BinG3, Th. 1] – cf. the more recent [Bin].

Theorem BG 3. *If $\varphi\in SE$ and $\varphi\uparrow\infty$, then U has a limiting moving average $K_U(x)=cx$ iff*

$$\frac{1}{\lambda(x)}\int_0^x U(y)d\lambda(y)\to c,$$

where $\lambda(x):=\varphi(x)\exp\tau_\varphi(x)$.

Corollary 5. *For $\varphi\in SE$ and $\varphi\uparrow\infty$, and with λ as previously, if F is of additive Beurling Π_φ-class with φ-index c, then*

$$\frac{1}{\lambda(x)}\int_0^x F(y)d\lambda(y)\to c.$$

9 Divided difference and double sweep

The concern of previous sections was the asymptotics of differences: $\Delta_t^\varphi h$ in the Beurling theory, and exceptionally in §8 moving averages $\Delta_t^\varphi h/\varphi$ in the Beurling version of the Bojanić-Karamata/de Haan theory. Introducing an appropriate general denominator ψ carries the same advantage as in BGT (e.g. 3.13.1) of 'double sweep': capturing the former theory via $\psi \equiv 1$ and the latter via $\psi \equiv \varphi$, embracing both through a common generalization – see Prop. 8′ above for a first hint of such possibilities. The work of this section is mostly to identify how earlier results generalize, much of it focussed on §3, to which we refer for group-theoretic notation; in particular $\mathbb{G}$ denotes the relevant *Popa group*, i.e. $\mathbb{G}^\rho$ for $\rho = \rho_\varphi$ for the appropriate φ, with $\mathbb{G}_+ := \{t : \eta_\rho(t) > 0\}$ its positive half-line.

Let $\varphi \in SE$; fix a φ-regularly varying $\psi > 0$ with φ-index γ and limit function g, i.e.

$$\psi(x + t\varphi(x))/\psi(x) \to g(t) \text{ loc. uniformly in } t \qquad (t > \rho^*), \qquad (G)$$

and, since $g(t)$ is a homomorphism (see Prop. 10 below), it is either $e^{\gamma t}$ ($\rho = 0$), or else $\eta_\rho(t)^\gamma$ (see [Ost4]). Recalling the notation $\Delta_t^\varphi h(x)$ from §1, we also write $\Delta_t^\varphi h/\psi(x)$ to mean $(\Delta_t^\varphi h(x))/\psi(x)$. We are concerned below with

$$H^*(t) := \limsup_x [\Delta_t^\varphi h/\psi], \qquad (H^*)$$

whenever this exists, and with the nature of the convergence. To specify whenever a case below of convergence arises, we write

$$\begin{aligned} H_+^*(t) &:= \lim_{\delta\downarrow 0} \limsup_{x\to\infty} \sup\{\Delta_s^\varphi h/\psi : s \in I_\delta^+(t)\}, \\ H_-^*(t) &:= \lim_{\delta\downarrow 0} \limsup_{x\to\infty} \sup\{\Delta_s^\varphi h/\psi : s \in I_\delta^-(t)\}, \\ H_\pm^*(t) &:= \lim_{\delta\downarrow 0} \limsup_{x\to\infty} \sup\{\Delta_s^\varphi h/\psi : s \in I_\delta(t)\}. \end{aligned}$$

We begin with an extension of Lemma 2, for which we recall the notation $\mathbb{A}_\Omega^\dagger$ of §8. The proofs are almost identical – so are omitted.

Lemma 2†. (i) *If $\varphi \in SE$ and (G) holds – then:*
(a) *if the convergence in (H^*) is uniform (resp. right-sidedly uniform) near $t = 0$, then it is uniform (resp. right-sidedly uniform) everywhere in $\mathbb{A}_\Omega^\dagger$ and for $u \in \mathbb{A}_\Omega^\dagger \cap \mathbb{R}_+$*

$$H_+^*(u) \leqslant H^*(u)g(u) + H_+^*(0);$$

(b) *if the convergence in* (H^*) *is uniform near* $t = u \in \mathbb{A}_\Omega^\dagger \cap (\mathbb{A}_\Omega^\dagger)_\circ^{-1} \cap \mathbb{R}_+$, *then it is uniform near* $t = 0$:

$$H_\pm^*(0) \leqslant H_\pm^*(u)g(u_\circ^{-1}) + H^*(u_\circ^{-1});$$

(ii) *if* $\rho = 0$ *and* $\varphi \in SN$ *is monotonic increasing and the convergence in* (H^*) *is right-sidedly uniform near* $t = u \in \mathbb{A}_\Omega^\dagger \cap \mathbb{R}_+$, *then it is right-sidedly uniform near* $t = 0$:

$$H_+^*(0) \leqslant H_+^*(u)g(u_\circ^{-1}) + H^*(u_\circ^{-1}).$$

For the next result, recall also the notation $\Omega_h^\dagger(t)$ of §8.

Proposition 10. (i) *With* g *as in* (G) *above,* $g(u \circ_\eta v) = g(u)g(v)$, *so that*

$$K_h(u \circ_\eta v \circ_\eta w) = K_h(u)g(v \circ_\eta w) + K_h(v)g(w) + K_h(w),$$

and furthermore

$$H^*(s \circ_\eta t) \leqslant H^*(t)g(s) + H^*(s) \qquad (s, t \in \mathbb{A}_\Omega^\dagger).$$

(ii) *If both of the following hold:*
(a) $H^*(t) > -\infty$ *for* t *in a subset* Σ *that is unbounded below;*
(b) *the Heiberg-Seneta condition* $\Omega_h^\dagger(0+) \leqslant 0$ *holds*
— *then* H^* *is finite on* $\mathbb{G}_+$ *and* $H^*(0+) = 0$.
Moreover, for $\mathbb{A}_\Omega^\dagger$ *dense in* $\mathbb{G}_+$,

$$H^*(u \circ_\eta v) = K(v)g(u) + H^*(u) \qquad (u \in \mathbb{G}_+^\rho, v \in \mathbb{A}^\varphi).$$

Proof. (i) The first assertion follows by writing $y = x \circ_\varphi u$ (as in Prop. 2(iii)) and taking limits in the identity

$$\psi(x \circ_\varphi (u \circ_\eta v))/\psi(x) = [\psi(y \circ_\varphi v/\eta_x(u)))/\psi(y)]\psi(x \circ_\varphi u)/\psi(x).$$

The assertion is a restatement of the Cauchy exponential equation for $e^{\gamma x}$ when $\rho = 0$ and for $\eta(x)^\gamma$ for $\rho > 0$, and so implies the second. As for the third assertion, argue as in Prop. 8′ above, but now with a new denominator $\psi(x_n)$.

(ii) The first assertion is proved from (a) as in [BinO12, Prop. 6], and the second from part (b) as in [BinO12, Prop. 8]; the latter uses part (i) and the two facts that $g(u \circ v) = g(u)g(v)$ and $g(u) \geqslant 1$ for $u > 0$. The second assertion is proved as in BGT Th. 3.2.5. □

As a corollary, since H^* is g-subadditive, we have the analogue of Th. 1 of [BinO12].

Theorem 8. *In the setting of Proposition 10, if $\mathbb{A}_\Omega^\dagger$ is dense, then $\mathbb{A}_{\Omega+}^\dagger = \mathbb{G}_+$ and for some $c, \gamma, \rho \in \mathbb{R}$:*
either (i) $\rho = 0$ *and* $H^*(u) \equiv cH_{(-\gamma)}(u) = c(1 - e^{-\gamma u})/\gamma$ $(u \geqslant 0)$,
or (ii) $\rho > 0$ *and* $H^*(u) \equiv [(1 + \rho u)^{\gamma+1} - 1]/[\rho(1 + \gamma)]$ $(u \geqslant 0)$.

Proof. As in Prop. 6 above, $(\mathbb{A}_\Omega^\dagger, \circ)$ is a subgroup. Now use Prop. 10, Theorem PJ, and Th. 3 of [BinO11]. □

10 Uniform Boundedness Theorem

As above, let h be Baire and $\varphi \in SE$ on $\mathbb{R}_+$ be positive. Thus for all divergent x_n (i.e. divergent to $+\infty$),

$$\varphi(x_n \circ_\varphi t)/\varphi(x_n) \to \eta(t) \text{ for all } t \geqslant 0 \text{ and } \varphi(x) = O(x).$$

So $y_n = x_n \circ_\varphi t = x_n(1 + t\varphi(x_n)/x_n)$ is divergent if x_n is.

We work additively, and recall that for $t > \rho^*$

$$H^*(t) := \limsup_{x\to\infty} h(x \circ_\varphi t) - h(x), \;\; H_*(t) := \liminf_{x\to\infty} h(x \circ_\varphi t) - h(x).$$

If $x_n \to \infty$ and $H^*(t) < \infty$, then for all large enough n

$$h(x_n \circ_\varphi t) - h(x_n) < n.$$

Likewise if $H_*(t) > -\infty$, then for all large enough n

$$h(x_n) - h(x_n \circ_\varphi t) < n.$$

In the theorem below we need to assume finiteness of both H^* and H_*; we recall that in the Karamata case, substituting y for $u + x$, one has

$$h^*(u) = \limsup[h(u + x) - h(x)] = -\liminf[h(y - u) - h(y)] = -h_*(-u).$$

This relationship is used implicitly in the standard development of the Karamata theory – see e.g. BGT, §2.1. Theorem 9 below extends [BinO8, Th. 8]. As the hypothesis is symmetric, the same proof yields the liminf case.

Theorem 9 (Uniform Boundedness Theorem; cf. [Ost2]). *For $\varphi \in SE$ and $\rho = \rho_\varphi$, suppose that $-\infty < H_*(t) \leqslant H^*(t) < \infty$ for $t \in S$ with $S \subseteq \mathbb{G}_+^\rho$ a non-meagre Baire set. Then for compact $K \subseteq S$*

$$\limsup_{x\to\infty}\left(\sup_{u\in K} h(x \circ_\varphi u) - h(x)\right) < \infty.$$

Proof. By compactness of K, it suffices to establish uniform boundedness locally at any point $u > \rho^*$. Suppose otherwise, and that this is witnessed by some $x_n \to \infty$ and $u_n \to u$. Writing $u_n := u + z_n$ with $z_n \to 0$ and passing if necessary from x_n to $\xi_n := x_n \circ_\varphi u$ (and using the identity $h(x_n \circ_\varphi u_n) - h(x_n) = [h(\xi_n \circ_\varphi z_n/\eta_x(u)) - h(\xi_n)] + [h(x_n \circ_\varphi u) - h(x_n)]$, where the first bracket tends to 0) w.l.o.g. we may assume $u = 0$, and

$$h(x_n \circ_\varphi z_n) - h(x_n) > 3n. \tag{3}$$

Put $y_n := x_n \circ_\varphi z_n$. As $\varphi \in SE$,

$$c_n := \varphi(x_n \circ_\varphi z_n)/\varphi(x_n) \to 1.$$

Write $\gamma_n(s) := c_n s + z_n$. Put

$$\begin{aligned} V_n :=& \ \ \{s \in \ \ S : h(x_n \circ_\varphi s) - h(x_n) < n\},\ H_k^+ := \textstyle\bigcap_{n\geqslant k} V_n, \\ W_n :=& \ \ \{s \in \ \ S : h(y_n) - h(y_n \circ_\varphi s) < n\},\ H_k^- := \textstyle\bigcap_{n\geqslant k} W_n. \end{aligned}$$

These are Baire sets, and since $-\infty < H_*(t) \leqslant H^*(t) < \infty$ on S,

$$S = \textstyle\bigcup_k H_k^+ = \bigcup_k H_k^-. \tag{4}$$

The increasing sequence of sets $\{H_k^+\}$ covers S. So for some k the set H_k^+ is non-negligible. Then, by (4), for some l the set

$$B := H_k^+ \cap H_l^-$$

is also non-negligible. Take $A := H_k^+$; then $B \subseteq H_l^-$ and $B \subseteq A$ with A, B non-negligible. Applying the Affine Two-sets Lemma [BinO10, Lemma

2] to the maps $\gamma_n(s) = c_n s + z_n$ with $c = \lim_n c_n = 1$, there exist $b \in B$ and an infinite set $\mathbb{M}$ with

$$\{c_m b + z_m : m \in \mathbb{M}\} \subseteq A = H_k^+.$$

That is, as $B \subseteq H_l^-$, there exist $t \in H_l^-$ and an infinite $\mathbb{M}_t$ with

$$\{\gamma_m(t) = c_m t + z_m : m \in \mathbb{M}_t\} \subseteq H_k^+.$$

In particular, for this t and $m \in \mathbb{M}_t$ with $m > k, l$,

$$t \in W_m \text{ and } \gamma_m(t) \in V_m.$$

As $\gamma_m(t) \in V_m$,

$$h(x_m \circ_\varphi \gamma_m(t)) - h(x_m) < m. \tag{5}$$

But $\gamma_m(t) = z_m + c_m t = z_m + t\varphi(y_m)/\varphi(x_m)$, so

$$x_m \circ_\varphi \gamma_m(t) = x_m + z_m\varphi(x_m) + t\varphi(y_m) = y_m \circ_\varphi t.$$

So, by (5),

$$h(y_m \circ_\varphi t) - h(x_m) < m.$$

But $t \in W_m$, so

$$h(y_m) - h(y_m \circ_\varphi t) < m.$$

Combining these with (4) and (3).

$$3m < h(y_m) - h(x_m) \leqslant \{h(y_m) - h(y_m \circ_\varphi t)\} + \{h(y_m \circ_\varphi t) - h(x_m)\} \leqslant 2m,$$

a contradiction. □

As in the classical Karamata case, this result implies global bounds on h – see BGT Th. 2.0.1.

Theorem 10. *In the setting of Theorem 9, for $\varphi \in SE$, if the set S on which $H^*(t)$ and $H_*(t)$ are finite contains a half-interval $[a_0, \infty)$ with $a_0 > 0$ — then there is a constant $C > 0$ such that for all large enough x and u*

$$h(u\varphi(x) + x) - h(x) \leqslant C \log u.$$

The proof parallels the end of the proof in BGT of Th. 2.0.1, but with the usual sequence of powers a^n replaced by a Popa-style generalization (cf. Prop. 2(v)):

$$a_{\varphi x}^{n+1} := a_{\varphi x}^n \circ_{\varphi x} a = a_{\varphi x}^n + a\eta_x^\varphi(a_{\varphi x}^n) \text{ with } a_{\varphi x}^1 = a.$$

It relies on estimation results for $a^m_{\varphi x}$ that are uniform in m (this only needs $\eta^\varphi_x \to \eta_\rho$ pointwise):[4]

Proposition 11. *If* $\varphi \in SE$ *with* $\rho = \rho_\varphi > 0$, *then for any* $a > 1$, $0 < \varepsilon < 1$,
(i)($a^m_{\varphi x}$-estimates under η^φ_x) *for all large enough* x:

$$(1-\varepsilon) \leqslant \eta^\varphi_x(a^m_{\varphi x})^{1/m}/\eta_\rho(a) \leqslant (1+\varepsilon), \qquad (m \in \mathbb{N});$$

(ii)($a^m_{\varphi x}$-estimates under η_ρ) *for all large enough* x:

$$\frac{\eta_\rho(a(1-\varepsilon))^m}{1-\varepsilon} - \frac{\varepsilon}{1-\varepsilon} \leqslant \eta_\rho(a^m_{\varphi x}) \leqslant \frac{\eta_\rho(a(1+\varepsilon))^m}{1+\varepsilon} + \frac{\varepsilon}{1+\varepsilon}, \qquad (m \in \mathbb{N});$$

(iii) $a^m_{\varphi x} \to \infty$; *and*
(iv) *there are* $C_\pm = C_\pm(\rho, a, \varepsilon) > 0$ *such that for all large enough* x *and* u:

$$a^m_{\varphi x} \leqslant u < a^{m+1}_{\varphi x} \Longrightarrow mC_- \leqslant \log u \leqslant (m+1)C_+.$$

11 Character degradation from limsup

We refer the reader to [BinO8] for a discussion, from the perspective of the practising analyst (employing 'naive' set theory), of the broader set-theoretic context below. For convenience we repeat part of the commentary in [BinO8]; for more detail see [BinO13]. As there so too here, our interest in the complexities induced by the *limsup* operation points us in the direction of definability and descriptive set theory, because of the question of whether certain specific sets, encountered in the course of the analysis, have the Baire property. The answer depends on what further axioms one admits. For us there are two alternatives yielding the kind of decidability we seek: Gödel's Axiom of Constructibility $V = L$, as an appropriate *strengthening* of the Axiom of Choice (AC) which creates definable sets without the Baire property (without measurability), or, at the opposite pole, the Axiom of Projective Determinacy, PD (see [MySw], or [Kec, 5.38.C]), an *alternative* to AC which guarantees the Baire property in the kind of definable sets we encounter. Thus to decide whether sets of the kind we encounter below have the Baire property, or are measurable, the answer is: it depends on the axioms of set theory that one

[4]See the Appendixfor the proofs of Th. 10 and Prop. 11.

adopts. It turns out that AC may be usefully weakened to the Axiom of Dependent Choice(s), DC; for details see [BinO13].

To formulate our results we need the language of descriptive set theory, for which see e.g. [JayR], [Kec], [Mos]. Within such an approach we will regard a function as a set, namely its *graph*; formulas written in naive set-theoretic notation then need a certain amount of formalization – for a quick approach to such matters refer to [Dra, Ch. 1,2] or the very brief discussion in [Kun, §1.2]. We need the beginning of the *projective hierarchy* in Euclidean space (see [Kec, S. 37.A]), in particular the following classes:

the *analytic* sets $\mathbf{\Sigma}^1_1$;

their complements, the *co-analytic* sets $\mathbf{\Pi}^1_1$;

the common part of the previous two classes, the *ambiguous* class $\mathbf{\Delta}^1_1 := \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$, that is, by Souslin's Theorem ([JayR, p. 5], and [MaKe, p.407] or [Kec, 14. C]) the *Borel* sets;

the *projections* (continuous images) of $\mathbf{\Pi}^1_1$ sets, forming the class $\mathbf{\Sigma}^1_2$;

their *complements*, forming the class $\mathbf{\Pi}^1_2$;

the *ambiguous* class $\mathbf{\Delta}^1_2 := \mathbf{\Sigma}^1_2 \cap \mathbf{\Pi}^1_2$;

and then: $\mathbf{\Sigma}^1_{n+1}$, the projections of $\mathbf{\Pi}^1_n$; their complements $\mathbf{\Pi}^1_{n+1}$; and the ambiguous class $\mathbf{\Delta}^1_{n+1} := \mathbf{\Sigma}^1_{n+1} \cap \mathbf{\Pi}^1_{n+1}$.

Throughout we shall be concerned with the cases $n = 1, 2$ or 3.

The notation reflects the fact that the canonical expression of the logical structure of their definitions, namely with the quantifiers (ranging over the reals, hence the superscript 1, as reals are type 1 objects – integers are of type 0) all at the front, is determined by a string of alternating quantifiers starting with an existential or universal quantifier (resp. $\mathbf{\Sigma}$ or $\mathbf{\Pi}$). Here the subscript accounts for the number of alternations.

Interest in the character of a function H is motivated by an interest within the theory of regular variation in the character of the level sets

$$H^k := \{s : |H(s)| < k\} = \{s : (\exists t)[(s,t) \in H \ \& \ |t| < k]\},$$

for $k \in \mathbb{N}$ (where as above H is identified with its graph). The set H^k is thus the projection of $H \cap (\mathbb{R} \times [-k, k])$ and hence is $\mathbf{\Sigma}^1_n$ if H is $\mathbf{\Sigma}^1_n$, e.g. it is $\mathbf{\Sigma}^1_1$, i.e. analytic, if H is analytic (in particular, Borel). Also

$$H^k = \{s : (\forall t)[(s,t) \in H \implies |t| \leqslant k]\} = \{s : (\forall t)[(s,t) \notin H \text{ or } |t| \leqslant k]\},$$

and so this is also $\mathbf{\Pi}^1_n$ if H is $\mathbf{\Sigma}^1_n$. Thus if H is $\mathbf{\Sigma}^1_n$ then H^k is $\mathbf{\Delta}^1_n$. So if $\mathbf{\Delta}^1_n$ sets are Baire, then for some k the set H^k is Baire non-null.

With this in mind, it suffices to consider upper limits; as before, we prefer to work with the additive formulation. Consider the definition:

$$H_{\varphi}^{*}(x) := \limsup_{t\to\infty}[h(t+x\varphi(t))-h(t)]. \tag{$**$}$$

Thus in general H_{φ}^{*} takes values in the extended real line. The problem is that the function H_{φ}^{*} is in general less well behaved than the function h – for example, if h is measurable/Baire, H_{φ}^{*} need not be. The problem we address here is the extent of this degradation – saying *exactly how much less regular* than h the limsup H_{φ}^{*} may be. The nub is the set S on which H_{φ}^{*} is finite. This set S is an additive semi-group on which the function H_{φ}^{*} is subadditive (see [BinO7]) – or additive, if limits exist (see[BinO6]). Furthermore, if H has Borel graph then H_{φ}^{*} has $\mathbf{\Delta}_2^1$ graph (see below). But in the presence of certain axioms of set-theory (for which see below) the $\mathbf{\Delta}_2^1$ sets have the Baire property and are measurable. Alternatively, if the $\mathbf{\Delta}_2^1$ character is witnessed by two $\mathbf{\Sigma}_2^1$ formulas Φ, Ψ such that the equivalence

$$\Phi(x) \Longleftrightarrow \neg\Psi(x)$$

is provable in ZF, i.e. *without reference* to AC, then A is said to be *provably* $\mathbf{\Delta}_2^1$. It then turns out that such sets are Baire/measurable – see [FenN]. So in such circumstances if S is large in either of these two senses, then in fact S contains a half-line.

The extent of the degradation in passing from h to H_{φ}^{*} is addressed in the following result, which we call the First Character Theorem, and then contrast it with two alternatives. These extend corresponding results established in the Karamata context in [BinO8] and differ from the former merely by duplicating assumptions previously made only on h there to identical ones on φ.

Theorem 11 (First Character Theorem). (i) *If h and φ are Borel (have Borel graph), then the graph of the function*

$$H^{*}(x) = \limsup\nolimits_{t\to\infty}[h(t+x\varphi(t))-h(t)]$$

is a difference of two analytic sets, hence is measurable and $\mathbf{\Delta}_2^1$. If the graphs of h and φ are $\mathcal{F}_\sigma$, then the graph of $H^{*}(x)$ is Borel.
(ii) *If h and φ are analytic (have analytic graph), then the graph of the function $H^{*}(x)$ is $\mathbf{\Pi}_2^1$.*

(iii) *If h and φ are co-analytic (have co-analytic graph), then the graph of the function $H^*(x)$ is $\mathbf{\Pi}^1_3$.*

The next two results assume much more, in requiring the existence of a limit (Th. 12) or a limit modulo an ultrafilter (Th. 13).

Theorem 12 (Second Character Theorem). *If the following limit exists:*

$$K_h(x) := \lim_{t\to\infty}[h(t + x\varphi(t)) - h(t)],$$

and $h, \varphi \in \mathbf{\Delta}^1_2$ – then the graph of K_h is $\mathbf{\Delta}^1_2$.

Theorem 13 (Third Character Theorem). *If the function h and the ultrafilter $\mathcal{U}$ (both on ω) are of class $\mathbf{\Delta}^1_2$ – then so is:*

$$K^{\mathcal{U}}_h h(t) := \mathcal{U}\text{-}\lim_n[h(n + t\varphi(n)) - h(n)].$$

The proofs of all three character theorems closely follow the proofs of the Karamata special case in [BinO8, §4], by using just two amendment procedures. Firstly, apply a *replacement rule:* all uses of the formula $y = h(x,t) := h(x + t) - h(t)$ (h as there) are to be replaced by a formalized conjunction of $y = h(x,s,t) := h(x + ts) - h(t)$ and $s = \varphi(x)$, as follows. Translate these two formulas to '$(x,s,t,y) \in h$ & $(x,s) \in \varphi$' (interpreting h and φ as naming the graphs of the two functions), and replace each $(x,t,y) \in h$ there by the the translate just indicated here above. Secondly, apply an *insertion rule*: insert the variable s everywhere to precede the variable w. An example of the translation will suffice; here is a sample amendment:

$$y = h(t + xs) - h(t) \Leftrightarrow (\exists u,v,s,w \in \mathbb{R})r(x,t,y,u,v,s,w),$$

where $r(x,t,y,u,v,s,w)$ stands for:

$$[\ y = u - v \ \&\ w = t + xs \ \&\ (w,u) \in h \ \&\ (t,v) \in h \ \&\ (x,s) \in \varphi]. \qquad (6)$$

Comment 1. In the first theorem (as also in [BinO8]) we deal with $H^*(x) = K^*_h(x) := \limsup_{t\to\infty} \Delta^\varphi_x h(t)$. The results are also true for $\limsup \Delta^\varphi_x h/\varphi(t)$ or $\limsup \Delta^\varphi_x h/\psi(t)$. The proofs are essentially the same; one needs the same assumptions on φ (or ψ) as on h.
Comment 2. The last of the three theorems applies under the assumption of Gödel's Axiom $V = L$ (see [Dev, §B.5, 453-489]), under which $\mathbf{\Delta^1_2}$ ultrafilters

exist on ω (e.g. for Ramsey ultrafilters – see [Z]). Above sets of natural numbers are identified with real numbers (via indicator functions), and so ultrafilters are subsets of $\mathbb{R}$ – for background see [CoN], or [HinS]. Th. 12 offers a midway position between the First and Second Character Theorems.

In Th. 13 $K_h^{\mathcal{U}} h(t)$ is additive, whereas in Th. 11 one has only sub-additivity (cf. BGT p. 62 equation (2.0.3)).

Comment 3. Replacing $h(n+t\varphi(n))-h(n)$ by $h(x(n)+t\varphi(x(n))-h(x(n))$, as in the Equivalence Theorem of [BinO3], to take limits along a specified sequence $\mathbf{x}:\omega\to\omega^\omega$, gives an 'effective' version of the character theorems – given an effective descriptive character of $\mathbf{x}$.

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Appendix: Global bounds

Below we need Bloom's [Blo] result that for x large enough the Beck sequence x_n^u defined recursively by its starting value x and the step-size u :

$$x_{n+1}^u = x_n \circ_\varphi u = x_n + u\varphi(x_n), \text{ with } x_0^u = x,\ x_1^u = x + u\varphi(x)$$

is divergent (see [BinO-B, §9] and compare [Ost-B, §6]). Say for $x \geqslant x_0$.

We briefly review a number of examples of Beck sequences; Example 2 is crucial.

Example 1. $a_\varphi^n = a_n^a$, (with a_n^a as above) so that $a_\varphi^{n+1} = a_\varphi^n \circ_\varphi a = a_\varphi^n + a\varphi(a_\varphi^n)$. Performing the recurrence the other way about, $u_{n+1} = u \circ_\varphi u_n = u + u_n\varphi(u)$ generates a GP:

$$u_n = (1 - \varphi(u)^{n+1}) \cdot u/(1 - \varphi(u)),$$

with

$$u_{n+1} - u_n = (u_n - u_{n-1})\varphi(u) = ... = u\varphi(u)^n.$$

For $\varphi \in GS$ the two are the same. They are not altogether dissimilar, as the other one has

$$a_\varphi^k = a[1 + \varphi(a) + \varphi(a_\varphi^2) + ... + \varphi(a_\varphi^{k-1})],$$

and, assuming divergence, the term-on-term growth is

$$\varphi(a_\varphi^k)/\varphi(a_\varphi^{k-1}) = \varphi(a_\varphi^{k-1} + a\varphi(a_\varphi^{k-1}))/\varphi(a_\varphi^{k-1}) \to \eta^\varphi(a),$$

so the *series* behaves, up to a multiplier $\varphi(a_\varphi^k)$, eventually like

$$\sum_{j<k} \eta^\varphi(a)^j = (1 - \eta^\varphi(a_\eta^k))/(1 - \eta^\varphi(a)).$$

Example 2. Consider the sequence

$$a_{\varphi x}^{n+1} := a_{\varphi x}^n \circ_{\varphi x} a = a_{\varphi x}^n + a\eta_x^\varphi(a_{\varphi x}^n) \text{ with } a_{\varphi x}^1 = a,$$

where a is fixed; on the back of Example 1 we guess that since uniformly in x

$$\eta_x^\varphi(a) \to \eta_\rho(a),$$

this $a^n_{\varphi x}$ is a divergent sequence for x large enough, say $x > x_a$. Indeed, it is – see the proof of Prop. 11; this is to be expected from the related iteration

$$a^{n+1}_\eta := a^n_\eta \circ_\eta a = a^n_\eta + a\eta_\rho(a^n_\eta) \text{ with } a^1_\eta = a,$$

where for $\rho = 0$ growth is linear: $\eta(a^n_\eta) = na$, whereas for $\rho > 0$ it is exponential:

$$\eta(a^n_\eta) = \eta(a^{n-1}_\eta \circ_\eta a) = \eta(a^{n-1}_\eta)\eta(a) = ... = \eta_\rho(a)^n = (1 + \rho a)^n.$$

Below we need the solution of a recurrence; we present this as a lemma, delaying the calculation to the end.

Lemma 3. *The solution of* $bv_{n+1} - v_n = r^n$ *for* $br \neq 1$ *is*

$$v_n = r^n/(br - 1) + b^{1-n}(v_1 - r/(br - 1)). \tag{soln}$$

If $b = \eta_\rho(a)$ *with* $\rho > 0$, $v_1 = 1/(\rho a)$, $r = 1 \pm \delta$, *with* $\delta = \varepsilon\rho a/\eta_\rho(a)$ *and* $0 < \varepsilon < 1$, *then*

$$v_1 - r/(br - 1) = \frac{\varepsilon/(\eta_\rho(a)\rho a)}{(1 + \varepsilon)} \text{ or} - \frac{\varepsilon/(\eta_\rho(a)\rho a)}{(1 - \varepsilon)}.$$

We now proceed to verify the details of Prop. 11 in §10.

Proof of Prop. 11. Fix $a, \rho > 0$ and $0 < \varepsilon < 1$. Taking $\delta := \varepsilon\rho a/\eta(a)$,

$$\eta(a) \pm \rho a\varepsilon = (1 + \rho a(1 \pm \varepsilon)) = \eta(a(1 \pm \varepsilon)) = \eta(a)(1 \pm \delta).$$

In particular, $\eta(a)(1 - \delta) = \eta(a(1 - \varepsilon)) > 1$, since $\varepsilon < 1$. Since $\eta_x(a) \to \eta(a)$, there is $X = X_{a,\varepsilon}$ with

$$|\eta(a) - \eta_x(a)| < \rho a\varepsilon : \qquad \eta(a)(1 - \delta) < \eta_x(a) < \eta(a)(1 + \delta) \qquad (x > X). \tag{δ-bd}$$

(i) By Prop. 2(v), for y_i running through $x \circ_\varphi a^{m-1}_{\varphi x}$, $x \circ_\varphi a^{m-2}_{\varphi x}, ..., x > X$,

$$\eta_x(a^m_{\varphi x}) = \prod\nolimits_{i=1}^m \eta_{y_i}(a), \tag{prod}$$

so that, by (δ-bd),

$$\eta(a(1 - \varepsilon)) \leqslant \eta_x(a^m_{\varphi x})^{1/m} \leqslant \eta(a(1 + \varepsilon)).$$

(ii) As $\eta \in GS$, $\eta(a^{n+1}_{\varphi x}) = \eta(a^n_{\varphi x} + a\eta_x(a^n_{\varphi x})) = \eta(a^n_{\varphi x})\eta(a\eta_x(a^n_{\varphi x})/\eta(a^n_{\varphi x}))$. So

$\eta(a^{n+1}_{\varphi x})/\eta(a^n_{\varphi x}) = 1 + \rho a\eta_x(a^n_{\varphi x})/\eta(a^n_{\varphi x}) : \qquad \eta(a^{n+1}_{\varphi x}) - \eta(a^n_{\varphi x}) = \rho a\eta_x(a^n_{\varphi x})$.

Putting $u_n := \eta(a^n_{\varphi x})/(\rho a\eta(a)^n)$, so that $u_1 = 1/(\rho a)$, and using (δ-bd) again,

$$(1-\delta)^n \leqslant \frac{\eta(a^{n+1}_{\varphi x}) - \eta(a^n_{\varphi x})}{\rho a\eta(a)^n} = \eta(a)u_{n+1} - u_n \leqslant (1+\delta)^n.$$

As $\eta(a)(1 \pm \delta) \neq 1$, apply Lemma 3 to $b = \eta(a)$ and $r = 1 \pm \delta$; then

$$\frac{(1-\delta)^n\eta(a)^n}{1-\varepsilon} - \frac{\varepsilon}{1-\varepsilon} \leqslant \eta(a^n_{\varphi x}) \leqslant \frac{(1+\delta)^n\eta(a)^n}{1+\varepsilon} + \frac{\varepsilon}{1+\varepsilon}.$$

(iii) As $\eta(a)(1-\delta) > 1$, the left inequality implies $a^m_{\varphi x}$ is *divergent.*
(iv) If $a^m_{\varphi x} \leqslant u < a^{m+1}_{\varphi x}$, then (as η_ρ is monotone), $\eta(a^m_{\varphi x}) \leqslant 1+\rho u \leqslant \eta(a^{m+1}_{\varphi x})$; so, for $x > X_e$

$$\frac{\eta(a(1-\varepsilon))^m}{(1-\varepsilon)} - \frac{1}{1-\varepsilon} \leqslant \rho u < \frac{\eta(a(1+\varepsilon))^{m+1}}{1+\varepsilon} - \frac{1}{1+\varepsilon}.$$

So for $\varepsilon < 1/2$

$$\frac{\eta(a(1-\varepsilon))^m}{(1-\varepsilon)} - 2 \leqslant \rho u < \frac{\eta(a(1+\varepsilon))^{m+1}}{1+\varepsilon}.$$

So for $u > 1$, as $2 + \rho u < (2+\rho)u$,

$$\frac{\eta(a(1-\varepsilon))^m}{(1-\varepsilon)(2+\rho)} \leqslant u < \frac{\eta(a(1+\varepsilon))^{m+1}}{\rho(1+\varepsilon)},$$

where $\log \eta(a(1-\varepsilon)) > 0$. Taking

$$C_-(\rho, a, \varepsilon) := \log \frac{\eta(a(1-\varepsilon))}{(\rho+2)(1-\varepsilon)}, \ C_+(\rho, a, \varepsilon) := \log \frac{\eta(a(1+\varepsilon))}{\rho(1+\varepsilon)},$$

$$mC_- \leqslant \log u < (m+1)C_+ \qquad (u \geqslant a > 1 \ \& \ x \geqslant X_{a,\varepsilon}). \qquad \square$$

We are now ready to prove Th. 9 of §10.

Proof of Theorem 10. (This parallels the tail end of the proof in BGT of Th. 2.0.1.) W.l.o.g. we assume that $\eta^{\varphi}(x) = 1 + \rho x$ with $\rho > 0$, as the case $\rho = 0$ is already known. By Theorem 8 (UBT) in §10, for any $a \geqslant a_0$

$$\limsup_{x\to\infty} \left(\sup_{a\leqslant u\leqslant 2a\eta(a)} h(x \circ_{\varphi} u) - h(x) \right) < \infty.$$

So there is C_a such that

$$\sup_{a\leqslant u\leqslant 2a\eta(a)} h(x \circ_{\varphi} u) - h(x) < C_a,$$

for all large enough x, say for $x > x_a$. Choose $a > \max\{a_0, x_a\}$.

As at the start of the proof of Prop. 11, but specializing to $\varepsilon = 1$, take $\delta := \rho a/\eta(a)$ to obtain (δ-bd) for $x > X$:

$$\eta(a)(1-\delta) < \eta_x(a) < \eta(a)(1+\delta) \qquad (x > X). \tag{**}$$

For $x > X$, fix $u \geqslant a = a^1_{\varphi x}$. Then, by Prop. 11(iii), we may choose $m = m_x(u)$ such that

$$a^{m-1}_{\varphi x} < a^m_{\varphi x} \leqslant u \leqslant a^{m+1}_{\varphi x}.$$

Now put $d := (u - a^{m-1}_{\varphi x})/\eta_x(a^{m-1}_{\varphi x})$, so that $u = a^{m-1}_{\varphi x} \circ_{\varphi x} d$; then

$$x \circ_{\varphi} u = [x + a^{m-1}_{\varphi x}\varphi(x)] + d\varphi(x + a^{m-1}_{\varphi x}\varphi(x)) = y \circ_{\varphi} d,$$

with $y = x \circ_{\varphi} a^{m-1}_{\varphi x}$; referring to $[a^m_{\varphi x} - a^{m-1}_{\varphi x}] + [a^{m+1}_{\varphi x} - a^m_{\varphi x}] = a\eta_x(a^{m-1}_{\varphi x}) + a\eta_x(a^m_{\varphi x})$ and to $u - a^{m-1}_{\varphi x} = d\eta_x(a^{m-1}_{\varphi x})$,

$$a\eta_x(a^{m-1}_{\varphi x}) \leqslant d\eta_x(a^{m-1}_{\varphi x}) < a\eta_x(a^{m-1}_{\varphi x}) + a\eta_x(a^m_{\varphi x}) = a\eta_x(a^{m-1}_{\varphi x}) + a\eta_y(a)\eta_x(a^{m-1}_{\varphi x}),$$

as in Prop. 2(v). But by (**) above, since $y \geqslant x > x_a$,

$$a \leqslant d < a(1 + \eta_y(a)) < a(1 + \eta(a)(1+\delta)) < a(1 + \rho a + \eta(a)) = 2a\eta(a),$$

as $\delta\eta(a) = \rho a$. So by choice of C_a,

$$h(x \circ_{\varphi} u) - h(x \circ_{\varphi} a^{m-1}_{\varphi x}) = h(y \circ_{\varphi} d) - h(y) < C_a,$$

as $d \in [a, 2a\eta(a)]$. As in Prop. 11,

$$x \circ_{\varphi} a^{n+1}_{\varphi x} = x \circ_{\varphi} (a^n_{\varphi x} \circ_{\varphi} a) = (x \circ_{\varphi} a^n_{\varphi x}) \circ_{\varphi} a,$$

and, setting $y_k = x \circ_\varphi a^k_{\varphi x}$ for $k = 0, ..., m-1$,

$$h(x \circ_\varphi a^{k+1}_{\varphi x}) - h(x \circ_\varphi a^k_{\varphi x}) = h((x \circ_\varphi a^k_{\varphi x}) \circ_\varphi a) - h(x \circ_\varphi a^k_{\varphi x}) = h(y_k \circ_\varphi a) - h(y_k) < C_a,$$

since $y_k \geqslant x > x_a$. So for $x > x_a$

$$h(x \circ_\varphi u) - h(x) = h(x \circ_\varphi u) - h(x \circ_\varphi a^{m-1}_{\varphi x}) + \sum_{k=1}^{m-1} (h(x \circ_\varphi a^k_{\varphi x}) - h(x \circ_\varphi a^{k-1}_{\varphi x})) < mC_a.$$

Again by Prop. 11, there is a constant C such that

$$m \leqslant C \log u.$$

Taking $K = C_a C$ yields the desired inequality. □

Proof of Lemma 3. A particular solution is $r^n/(br-1)$, $bw_{n+1} - w_n = 0$ for $w_n = v_n - r^n/(br-1)$ and $w_n = w_1 b^{1-n}$, where $w_1 = v_1 - r/(br-1)$.

For $b = \eta_\rho(a)$, $v_1 = 1/(\rho a)$ and $r = 1 \pm \delta$, we calculate that

$$\begin{aligned} \rho a w_1 &= \frac{[\eta(a)(1 \pm \delta) - 1] - \rho a(1 \pm \delta)}{(1 + \rho a)(1 \pm \delta) - 1} = \frac{[(1 + \rho a)(1 \pm \delta) - 1] - \rho a(1 \pm \delta)}{\rho a + \eta(a)(\pm \delta)} \\ &= \pm \frac{\delta}{\rho a + \eta(a)(\pm \delta)} = \pm \frac{\varepsilon/\eta(a)}{(1 + (\pm 1)\varepsilon)} = \frac{\varepsilon/\eta(a)}{(1 + \varepsilon)}, \text{ or } - \frac{\varepsilon/\eta(a)}{(1 - \varepsilon)} \; (-) \; . \end{aligned}$$

We close with

Proof of the Characterization Theorem (Uniform semicontinuity)..
In the notation above, for $n > m$

$$f(t) - \varepsilon \leqslant \inf\{f_n(s) : s \in I_\delta(t)\} \leqslant \sup\{f_n(s) : s \in I_\delta(t)\} \leqslant f(t) + \varepsilon.$$

So

$$f(t) - \varepsilon \leqslant \liminf_n \inf\{f_n(s) : s \in I_\delta(t)\} \leqslant \limsup_n \sup\{f_n(s) : s \in I_\delta(t)\} \leqslant f(t) + \varepsilon.$$

We my now take limits as $\delta \downarrow 0$ to obtain

$$f(t) - \varepsilon \leqslant \lim_{\delta \downarrow 0} \liminf_n \inf\{f_n(s) : s \in I_\delta(t)\} \leqslant \lim_{\delta \downarrow 0} \limsup_n \sup\{f_n(s) : s \in I_\delta(t)\} \leqslant f(t) + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary,

$$\begin{aligned} f(t) &= \lim_{\delta \downarrow 0} \limsup_n \sup\{f_n(s) : s \in I_\delta(t)\} \\ &= \lim_{\delta \downarrow 0} \liminf_n \inf\{f_n(s) : s \in I_\delta(t)\}. \end{aligned}$$

Now suppose that $f(t) = \lim_{\delta \downarrow 0} \limsup_n \sup\{f_n(s) : s \in I_\delta(t)\}$ and $\varepsilon > 0$. Then for some $\delta > 0$

$$\limsup_n \sup\{f_n(s) : s \in I_\delta(t)\} < f(t) + \varepsilon,$$

and so there is N_t such that for $n > N_t$

$$\sup\{f_n(s) : s \in I_\delta(t)\} < f(t) + \varepsilon$$

and so

$$f_n(s) < f(t) + \varepsilon \text{ for } n > N_t \text{ and } s \in I_\delta(t).$$

By a similar argument there is δ' and N_t' so that

$$f_n(s) > f(t) - \varepsilon \text{ for } n > N_t' \text{ and } s \in I_{\delta'}(t). \ \Box$$