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Beurling slow and regular variation

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Dedicated to August Aimé (Guus) Balkema and Paul Embrechts

Abstract

We give a new theory of Beurling regular variation (Part II). This includes the previously known theory of Beurling slow variation (Part I) to which we contribute by extending Bloom's theorem. Beurling slow variation arose in the classical theory of Karamata slow and regular variation. We show that the Beurling theory includes the Karamata theory.

Contents

Introduction

Beurling slow variation (already known) and Beurling regular variation (new here) are related to the classical Karamata (or ordinary) slow and regular variation. This is the study of limit relations of the form

$$f(tx)/f(x) \longrightarrow g(t) \quad \text{as } x \longrightarrow \infty,\ \forall t \in \mathbb{R}_+, \tag{K}$$

written multiplicatively, or

$$h(u+x) - h(x) \longrightarrow k(u) \quad \text{as } x \longrightarrow \infty,\ \forall u \in \mathbb{R}, \tag{K_+}$$

written additively. For background on the extensive theory and applications here, see [9] (BGT below). As we show (see § 10.3), the Beurling theory here includes the Karamata theory.

One early success of Karamata regular variation was in Wiener Tauberian theory. Here the basic result is the following theorem.

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Theorem W (Wiener's Tauberian theorem). *For $K \in L_1(\mathbb{R})$ with the Fourier transform $\hat{K}$ of K non-vanishing on $\mathbb{R}$, and $H \in L_\infty(\mathbb{R})$: if*

$$\int K(x-y)H(y)\,dy \longrightarrow c\int K(y)\,dy \quad (x \longrightarrow \infty),$$

then, for all $G \in L_1(\mathbb{R})$,

$$\int G(x-y)H(y)\,dy \longrightarrow c\int G(y)\,dy \quad (x \longrightarrow \infty).$$

There are corresponding multiplicative versions with $\int_0^\infty K(x/y)H(y)\,dy/y$, etc. The Wiener Tauberian theory adapts well to most of the classical special summability methods (see, for example, Hardy [**41**, Chapter XII] and Korevaar [**45**, II, IV, VI]), with the exception of the vitally important *Borel method* $(s_n) \mapsto \sum_0^\infty s_k\, e^{-x}x^k/k!$. This was known to be closely linked to the *Valiron method* which maps $(s_n) \mapsto \sum_0^\infty s_k \cdot \exp\{-\frac{1}{2}(x-k)^2/x\}/\sqrt{2\pi x}$. Beurling observed (in unpublished lectures, see § 10.1 for references) that both may be reduced to the Wiener case in Theorem W by exploiting the fact that $\varphi(x) := \sqrt{x}$ has the property of *Beurling slow variation*; see Part I. This is based on a Uniform Convergence Theorem (UCT), extending a result of Bloom's concerning continuous functions to functions with the Darboux property that are measurable or have the Baire property (*Baire* below).

The roots of *Beurling regular variation* lie in a variant of Karamata regular variation, Bojanic-Karamata/de *Haan theory* (BGT, Ch. III, [**24**]), which goes back to [**23**]. The Bojanic-Karamata/*de Haan* Π-*class* is an important subclass of the class of Karamata slowly varying functions. These functions are increasing: their (rapidly increasing) inverses form the *de Haan* Γ-*class*. This in turn motivates our theory of *Beurling regular variation* developed in Part II.

For further background and more references, see our arXiv papers [**17**, **18**] and [**19**].

Despite the numerous positive results below, the question that motivated this study remains open, at least in its original form. Does Bloom's theorem/UCT extend to measurable/Baire functions, that is, can one omit the Darboux requirement? Does it even extend to Baire-1 functions?

Part I. Beurling slow variation

1. *Bloom's theorem*

We say that $\varphi > 0$ is *Beurling slowly varying*, $\varphi \in \mathrm{BSV}$, if $\varphi(x) = o(x)$ as $x \to \infty$ and

$$\varphi(x + t\varphi(x))/\varphi(x) \longrightarrow 1 \quad (x \longrightarrow \infty) \quad \forall t. \tag{BSV}$$

Using the additive notation $h := \log\varphi$ (whenever convenient):

$$h(x + t\varphi(x)) - h(x) \longrightarrow 0 \quad (x \longrightarrow \infty) \quad \forall t. \tag{BSV$_+$}$$

If (as in the UCT for Karamata slow variation) the convergence here is locally uniform in t, then we say that φ is *self-neglecting*, $\varphi \in \mathrm{SN}$; we write (SN) for the corresponding strengthening of (BSV). For applications, see BGT § 2.11 and [**3**, 6.7]. See also § 10.3 for an extension of SN to the class SE of *self-equivarying functions*.

We may now state Beurling's extension to Wiener's Tauberian theorem (for background and further results, see § 10.1 and [**19**, **45**]).

Theorem (Beurling's Tauberian theorem). *If $\varphi \in \mathrm{BSV}$, $K \in L_1(\mathbb{R})$ with $\hat{K}$ non-zero on $\mathbb{R}$, H is bounded, and*

$$\int K\left(\frac{x-y}{\varphi(x)}\right) H(y)\,dy/\varphi(x) \longrightarrow c\int K(y)\,dy \quad (x \longrightarrow \infty),$$

then, for all $G \in L_1(\mathbb{R})$,

$$\int G\left(\frac{x-y}{\varphi(x)}\right) H(y)\, dy/\varphi(x) \longrightarrow c \int G(y)\, dy \quad (x \longrightarrow \infty).$$

Note that the arguments of K and G here involve *both* the additive group operation on the line *and* the multiplicative group operation on the half-line. Thus Beurling's Tauberian theorem, although closely related to Wiener's (which it contains, as the case $\varphi \equiv 1$), is structurally different from it. One may also see here the relevance of the *affine group*, $\mathcal{A}$*ff*, already well used for regular variation (see, for example, BGT § 8.5.1 and §§ 3, 5).

Analogously to Karamata's UCT, the following result was proved by Bloom in 1976 [**21**] (an extended and simplified version is in BGT, Theorem 2.11.1).

THEOREM (Bloom's theorem). *If* $\varphi \in$ BSV *with* φ *continuous, then* $\varphi \in$ SN: (BSV) *holds locally uniformly.*

As above, our motivating question here is whether one can extend this to φ measurable and/or Baire; see BGT § 2.11, [**45**, IV.11] for textbook accounts. We give below a number of results in this direction. Our methods involve tools from infinite combinatorics, and replacement of quantitative measure theory by qualitative measure theory. We also prove a representation theorem (Theorem 8), whence any $\varphi \in$ SN satisfies $\varphi(x) = o(x)$, so $\varphi \in$ BSV.

2. *Monotone functions*

We suggest that the reader cast his eye over the proof of Bloom's theorem, in either [**21**] or BGT § 2.11, it is quite short. Like most proofs of the UCT for Karamata slow variation, it proceeds by contradiction, assuming that the desired uniformity fails, and working with two sequences, $t_n \in [-T, T]$ and $x_n \to \infty$, witnessing to its failure.

The next result, in which we assume that φ monotone (φ increasing to infinity is the only case that requires proof) is quite simple. But it is worth stating explicitly, for three reasons.

(1) It is a complement to Bloom's theorem, and to the best of our knowledge the first new result in the area since 1976.

(2) The case of φ increasing is by far the most important one for applications. For, taking G the indicator function of an interval in Beurling's Tauberian theorem, the conclusion there has the form of a *moving average*:

$$\frac{1}{a\varphi(x)} \int_x^{x+a\varphi(x)} H(y)\, dy \longrightarrow c \quad (x \longrightarrow \infty)\ \forall a\ > 0.$$

Such moving averages are *Riesz* (*typical*) *means*, and here φ increasing to ∞ is natural in context. For a textbook account, see [**30**] and the recent [**19**]; for applications, in analysis and probability theory, see [**8**, **20**]. The prototypical case is $\varphi(x) = x^\alpha$ $(0 < \alpha < 1)$; this corresponds to $X \in L_{1/\alpha}$ for the probability law of X.

(3) Theorem 1 is closely akin to results of de Haan on the Gumbel law Λ in extreme-value theory; see BGT §§ 3.10, 8.13.

We offer three proofs (two here and a third after Theorem 2M in § 4) of the result, as each is short and illuminating in its own way.

For the first, recall that if a sequence of monotone functions converges point-wise to a continuous limit, the convergence is uniform on compact sets. See, for example, Pólya and Szegő [**62**, Vol. 1, p. 63, 225, Problems II 126, 127] and Boas [**22**, § 17, pp. 104–105]. (The proof is a simple compactness argument, complementing the better-known result of Dini, in which it is the convergence, rather than the functions, that is monotone; see, for example, [**64**, 7.13].)

THEOREM 1 (Monotone Beurling UCT). *If* $\varphi \in \mathrm{BSV}$ *is monotone, then* $\varphi \in \mathrm{SN}$*: the convergence in* (BSV) *is locally uniform.*

First proof. As in [**21**] or BGT § 2.11, we proceed by contradiction. Pick $T > 0$, and assume that the convergence is not uniform on $[-T, 0]$ (the case $[0, T]$ is similar). Then there exist $\varepsilon_0 \in (0, 1)$, $t_n \in [-T, 0]$ and $x_n \to \infty$ such that

$$|\varphi(x_n + t_n\varphi(x_n))/\varphi(x_n) - 1| \geqslant \varepsilon_0 \quad \forall n.$$

Write

$$f_n(t) := \varphi(x_n + t\varphi(x_n))/\varphi(x_n) - 1.$$

Then f_n is monotone, and tends point-wise to 0 by (BSV). So, by the Pólya–Szegő result above, the convergence is uniform on compact sets. This contradicts $|f_n(t_n)| \geqslant \varepsilon_0$ for all n. □

The second proof is based on the following result, thematic for the approach followed in § 4. We need some notation that will also be of use later. Below, $x > 0$ will be a continuous variable, or a sequence $x := \{x_n\}$ diverging to $+\infty$ (briefly, *divergent sequence*), according to context. We put

$$V_n^x(\varepsilon) := \{t \geqslant 0 : |\varphi(x_n + t\varphi(x_n))/\varphi(x_n) - 1| \leqslant \varepsilon\}, \quad H_k(\varepsilon) := \bigcap_{n \geqslant k} V_n^x(\varepsilon).$$

LEMMA 1. *For* $\varphi > 0$ *monotonic increasing and* $\{x_n\}$ *a divergent sequence, each set* $V_n^x(\varepsilon)$*, and so also each set* $H_k^x(\varepsilon)$*, is an interval containing* 0.

Proof. Since $x + s\varphi(x) > x$ for $s > 0$, one has $1 \leqslant \varphi(x + t\varphi(x))/\varphi(x)$. Also if $0 < s < t$, then, as $\varphi(x) > 0$, one has $x + s\varphi(x) < x + t\varphi(x)$. So, if $t \in V_n^x(\varepsilon)$, then

$$1 \leqslant \varphi(x_n + s\varphi(x_n))/\varphi(x_n) \leqslant \varphi(x_n + t\varphi(x_n))/\varphi(x_n) \leqslant 1 + \varepsilon,$$

and so $s \in V_n^x(\varepsilon)$. The remaining assertions now follow, because an intersection of intervals containing 0 is an interval containing 0. □

Second proof of Theorem 1. Suppose otherwise; then there are $\varepsilon_0 > 0$ and sequences $x_n := x(n) \to \infty$ and $u_n \to u_0$ such that

$$|\varphi(x_n + u_n\varphi(x_n))/\varphi(x_n) - 1| > \varepsilon_0 \quad (\forall n \in \mathbb{N}).$$

Since φ is Beurling slowly varying, the increasing sets $H_k^x(\varepsilon_0)$ cover $\mathbb{R}_+$ and so, being increasing intervals (by Lemma 1), their interiors cover the compact set $K := \{u_n : n = 0, 1, 2, \ldots\}$. So, for some integer k, the set $H_k^x(\varepsilon_0)$ already covers K, and then so does $V_k^x(\varepsilon_0)$. But this implies that

$$|\varphi(x_k + u_k\varphi(x_k))/\varphi(x_k) - 1| \leqslant \varepsilon_0,$$

contradicting the above at $n = k$. □

Remark. Of course, the uniformity property of φ is equivalent to the sets $H^x_k(\varepsilon)$ containing arbitrarily large intervals $[0,t]$ for large enough k (for all divergent $\{x_n\}$).

3. *Infinite combinatorics*

As usual with proofs involving regular variation the nub lies in infinite combinatorics, to which we now turn. We recall that one can handle Baire and measurable cases together by working bi-topologically, using the Euclidean topology in the Baire case (the primary case) and the density topology in the measure case; see § 10.2. The *negligible* sets are the meagre sets in the Baire case and the null sets in the measure case; we say that a property holds *quasi everywhere* if it holds off a negligible set.

We work in the affine group $\mathcal{A}\!f\!f$ acting on $(\mathbb{R},+)$ using the notation

$$\gamma_n(t) = c_n t + z_n,$$

where $c_n \to c_0 = c > 0$ and $z_n \to 0$ as $n \to \infty$. These are to be viewed as (self-) homeomorphisms of $\mathbb{R}$ under either $\mathcal{E}$, the Euclidean topology, or $\mathcal{D}$, the Density topology. Recall that the open sets of $\mathcal{D}$ are measurable subsets, all points of which are (Lebesgue) density points, and that (i) Baire sets under $\mathcal{D}$ are precisely the Lebesgue measurable sets, (ii) the nowhere dense sets of $\mathcal{D}$ are precisely the null sets and (iii) Baire's Theorem holds for $\mathcal{D}$. (See Kechris [**43**, 17.47].) We recall the following definition and theorem from [**12**], which we apply taking the space X to be $\mathbb{R}$ with one of $\mathcal{E}$ or $\mathcal{D}$.

Definition. A sequence of homeomorphisms $h_n : X \to X$ satisfies the *weak category convergence* condition (wcc) if:

For any non-meagre open set $U \subseteq X$, there is a non-meagre open set $V \subseteq U$ such that, for each $k \in \mathbb{N}$,

$$\bigcap_{n \geqslant k} V \backslash h_n^{-1}(V) \text{ is meagre.}$$

Theorem CET (Category Embedding Theorem). *Let X be a topological space and $h_n : X \to X$ be homeomorphisms satisfying* (*wcc*)*. Then, for any Baire set T, for quasi-all $t \in T$ there is an infinite set $\mathbb{M}_t \subseteq \mathbb{N}$ such that*

$$\{h_m(t) : m \in \mathbb{M}_t\} \subseteq T.$$

From here, we deduce the following lemma.

Lemma 2 (Affine Two-sets Lemma). *For $c_n \to c > 0$ and $z_n \to 0$, if $cB \subseteq A$ for A, B non-negligible* (*measurable/Baire*)*, then, for quasi-all $b \in B$, there exists an infinite set $\mathbb{M} = \mathbb{M}_b \subseteq \mathbb{N}$ such that*

$$\{\gamma_m(b) = c_m b + z_m : m \in \mathbb{M}\} \subseteq A.$$

Proof. It is enough to prove the existence of one such point b, as the Generic Dichotomy Principle (for which see [**14**, Theorem 3.3]) applies here, because we may prove the existence of such a b in any non-negligible $\mathcal{G}_\delta$-subset B' of B, by replacing B below with B'. (One checks that the set of bs with the desired property is Baire, and so its complement in B cannot contain a non-negligible $\mathcal{G}_\delta$.)

Writing $T := cB$ and $w_n = c_n c^{-1}$, so that $c_n = w_n c$ and $w_n \to 1$, put

$$h_n(t) := w_n t + z_n.$$

Then h_n converges to the identity in the supremum metric, so (wcc) holds by [13, Theorem 6.2] (First Verification Theorem), and so Theorem CET above applies for the Euclidean case; applicability in the measure case is established as [11, Corollary 4.1]. (This is the basis on which the affine group preserves negligibility.) So there are $t \in T$ and an infinite set of integers $\mathbb{M}$ with

$$\{w_m t + z_m : m \in \mathbb{M}\} \subseteq T.$$

But $t = cb$ for some $b \in B$ and so, as $w_m c = c_m$, one has

$$\{c_m b + z_m : m \in \mathbb{M}\} \subseteq cB \subseteq A. \qquad \square$$

4. *Darboux property*

Here we generalize Bloom's Theorem (from continuity to the Darboux property) and simplify his proof.

We recall the *Darboux property*, also called the *intermediate value property*, that if a (real-valued) function attains two values in an interval then it must attain all intermediate values in that interval. Bloom uses continuity only through the Darboux property. It is much weaker than continuity, it does not imply measurability, nor the Baire property. For measurability, see the papers of Halperin [39, 40]; for the Baire property, see, for example, [63] and also § 10.2. Conversely, neither measurability nor the Baire property implies the Darboux property, as $\mathbf{1}_{\mathbb{Q}}$ shows.

We use Lemma 2 to prove Theorem 4B, which implies Bloom's Theorem, as continuous functions are Baire and Darboux. We note a result of Kuratowski and Sierpiński [48] that, for a function of Baire class 1 (for which see below), the Darboux property is equivalent to its graph being connected; so Theorem 4 goes beyond the class of functions considered by Bloom.

We begin with some infinite combinatorics associated with a positive function $\varphi \in \mathrm{BSV}$.

Definitions. Say that $\{u_n\}$ with limit u is a *witness sequence at* u (for non-uniformity in φ) if there are $\varepsilon_0 > 0$ and a divergent sequence x_n such that, for $h = \log \varphi$,

$$|h(x_n + u_n \varphi(x_n)) - h(x_n)| > \varepsilon_0 \quad \forall n \in \mathbb{N}. \tag{1}$$

Say that $\{u_n\}$ with limit u is a *divergent witness sequence* if also

$$h(x_n + u_n \varphi(x_n)) - h(x_n) \longrightarrow \pm\infty.$$

Thus a divergent witness sequence is a special type of witness sequence, but, as we show, it is these that characterize the absence of uniformity in BSV.

We begin with a lemma that yields simplifications later; it implies a Beurling analogue of the Bounded Equivalence Principle in the Karamata theory, first noted in [10]. As it shifts attention to the origin, we call it the Shift Lemma of BMA. Below *uniform near* a point u means 'uniformly on sequences converging to u' and is equivalent to local uniformity at u (that is, on compact neighbourhoods of u).

Lemma 3 (Shift Lemma). *For any u, convergence in* (BSV_+) *is uniform near $t = 0$ if and only if it is uniform near $t = u$.*

Proof. Take $z_n \to 0$. For any u write $y_n := x_n + u\varphi(x_n)$, $\gamma_n := \varphi(x_n)/\varphi(y_n)$ and $w_n = \gamma_n z_n$. Then $\gamma_n \to 1$, so $w_n \to 0$. But

$$\begin{aligned} &h(x_n + (u+z_n)\varphi(x_n)) - h(x_n) \\ &\quad = [h(x_n + u\varphi(x_n) + z_n\varphi(x_n)) - h(x_n + u\varphi(x_n))] + [h(x_n + u\varphi(x_n)) - h(x_n)] \\ &\quad = [h(y_n + z_n\gamma_n\varphi(y_n)) - h(y_n)] + [h(y_n) - h(x_n)], \end{aligned}$$

that is,

$$h(x_n + (u+z_n)\varphi(x_n)) - h(x_n) = [h(y_n + w_n\varphi(y_n)) - h(y_n)] + [h(y_n) - h(x_n)].$$

The result follows, since $h(y_n) - h(x_n) \to 0$, as $\varphi \in \mathrm{BSV}$ and $y_n \to \infty$. □

THEOREM 2B (Divergence Theorem, Baire version). *If $\varphi \in \mathrm{BSV}$ has the Baire property and u_n with limit u is a witness sequence, then u_n is a divergent witness sequence.*

Proof. As u_n is a witness sequence, for some $x_n \to \infty$ and $\varepsilon_0 > 0$ one has (1), with $h = \log\varphi$, as always. By the Shift Lemma (Lemma 3), we may assume that $u = 0$. So (as in the proof of Lemma 2) we will write z_n for u_n. If z_n is not a divergent witness sequence, then $\{\varphi(x_n+z_n\varphi(x_n))/\varphi(x_n)\}$ contains a bounded subsequence and so a convergent sequence. Without loss of generality, we thus also have

$$c_n := \varphi(x_n + z_n\varphi(x_n))/\varphi(x_n) \longrightarrow c \in (0,\infty). \tag{2}$$

Write $\gamma_n(s) := c_n s + z_n$ and $y_n := x_n + z_n\varphi(x_n)$. Then $y_n = x_n(1 + z_n\varphi(x_n)/x_n) \to \infty$ and

$$|h(y_n) - h(x_n)| \geqslant \varepsilon_0. \tag{3}$$

Now take $\eta = \varepsilon_0/3$ and amend the notation of § 2 to read

$$V_n^x(\eta) := \{s \geqslant 0 : |h(x_n + s\varphi(x_n)) - h(x_n)| \leqslant \eta\}, \quad H_k^x(\eta) := \bigcap_{n \geqslant k} V_n^x(\eta).$$

These are Baire sets, and

$$\mathbb{R} = \bigcup_k H_k^x(\eta) = \bigcup_k H_k^y(\eta), \tag{4}$$

as $\varphi \in \mathrm{BSV}$. The increasing sequence of sets $\{H_k^x(\eta)\}$ covers $\mathbb{R}$. So for some k, the set $H_k^x(\eta)$ is non-negligible. Furthermore, as $c > 0$, the set $c^{-1}H_k^x(\eta)$ is non-negligible and so, by (4), for some l the set

$$B := (c^{-1}H_k^x(\eta)) \cap H_l^y(\eta)$$

is also non-negligible. Taking $A := H_k^x(\eta)$, one has $B \subseteq H_l^y(\eta)$ and $cB \subseteq A$ with A, B non-negligible. Applying Lemma 2 to the maps $\gamma_n(s) = c_n s + z_n$, there exist $b \in B$ and an infinite set $\mathbb{M}$ such that

$$\{c_m b + z_m : m \in \mathbb{M}\} \subseteq A = H_k^x(\eta).$$

That is, as $B \subseteq H_l^y(\eta)$, there exist $t \in H_l^y(\eta)$ and an infinite $\mathbb{M}_t$ such that

$$\{\gamma_m(t) = c_m t + z_m : m \in \mathbb{M}_t\} \subseteq H_k^x(\eta).$$

In particular, for this t and $m \in \mathbb{M}_t$ with $m > k, l$ one has

$$t \in V_m^y(\eta) \quad \text{and} \quad \gamma_m(t) \in V_m^x(\eta).$$

Fix such an m. As $\gamma_m(t) \in V_m^x(\eta)$,

$$|h(x_m + \gamma_m(t)\varphi(x_m)) - h(x_m)| \leqslant \eta. \tag{5}$$

But $\gamma_m(t) = c_m t + z_m = z_m + t\varphi(y_m)/\varphi(x_m)$, so

$$x_m + \gamma_m(t)\varphi(x_m) = x_m + z_m\varphi(x_m) + t\varphi(y_m) = y_m + t\varphi(y_m),$$

'absorbing' the affine shift $\gamma_m(t)$ into y. So, by (5),

$$|h(y_m + t\varphi(y_m)) - h(x_m)| \leqslant \eta.$$

But $t \in V_m^y(\eta)$, so

$$|h(y_m + t\varphi(y_m)) - h(y_m)| \leqslant \eta.$$

Combining,

$$\begin{aligned} |h(y_m) - h(x_m)| &\leqslant |h(y_m + t\varphi(y_m)) - h(y_m)| + |h(y_m + t\varphi(y_m)) - h(x_m)| \\ &\leqslant 2\eta < \varepsilon_0, \end{aligned}$$

contradicting (3). □

THEOREM 2M (Divergence Theorem, Measure version). *If $\varphi \in$ BSV is measurable and u_n with limit u is a witness sequence, then u_n is a divergent witness sequence.*

Proof. The argument above applies, with $\mathcal{D}$ in place of $\mathcal{E}$. □

As an immediate corollary, we have the following.

Third proof of Theorem 1. If not, then there exists a witness sequence u_n with limit u. By Lemma 3, without loss of generality $u > 0$. Let $v > u > w > 0$. Since $\varphi \in$ BSV,

$$\varphi(x_n + v\varphi(x_n))/\varphi(x_n) \longrightarrow 1 \quad \text{and} \quad \varphi(x_n + w\varphi(x_n))/\varphi(x_n) \longrightarrow 1,$$

so there is N such that both $(1/2)\varphi(x_n) < \varphi(x_n + w\varphi(x_n))$ and $\varphi(x_n + v\varphi(x_n)) < 2\varphi(x_n)$ for all $n > N$. By increasing N if necessary, we may take $w < u_n < v$ for $n > N$. But then

$$(\tfrac{1}{2})\varphi(x_n) < \varphi(x_n + w\varphi(x_n)) < \varphi(x_n + u_n\varphi(x_n)) < \varphi(x_n + v\varphi(x_n)) < 2\varphi(x_n)$$

implies that $\frac{1}{2} < \varphi(x_n + u_n\varphi(x_n))/\varphi(x_n) < 2$, contradicting Theorem 2B/2M, as φ is Baire/measurable. □

We also have, as a further corollary, the following theorem.

THEOREM 3. *For Baire/measurable $\varphi \in$ BSV, if either $\varphi(x)$ has bounded range (bounded away from 0 and infinity), or $\varphi(x)/x$ is bounded away from 0, then $\varphi \in$ SN : (BSV) holds locally uniformly.*

Proof. Suppose otherwise. As above, by Lemma 3 and Theorem 2, there exists a divergent witness sequence $z_n \to 0$ such that, for some $x_n \to \infty$ and $\varepsilon_0 > 0$, the inequality (1) holds with $u_n = z_n$ and $y_n := x_n + z_n\varphi(x_n)$. Without loss of generality, we may assume that $y_n > 0$ and $|z_n| \leqslant 1$ for all n.

Suppose first that $0 < K < \varphi(x) < L$ for all x; then $0 < K/L < \varphi(y_n)/\varphi(x_n) < L/K$ for all n, and so the witness sequence is not divergent, which is a contradiction.

Next, suppose that $0 < K < \varphi(x)/x < L$ for all x (possible as $\varphi(x) = o(x)$). Without loss of generality, we may now also assume that $|z_n| < 1/2L$ for all n and so $|z_n\varphi(x_n)/x_n| < \frac{1}{2}$. Then

$$0 < \frac{K}{2L} < \frac{\varphi(y_n)}{\varphi(x_n)} < \frac{L(1+L)}{K},$$

for all n, since

$$\frac{\varphi(y_n)}{\varphi(x_n)} = \frac{\varphi(y_n)}{y_n}\frac{y_n}{x_n}\frac{x_n}{\varphi(x_n)} = \frac{\varphi(y_n)}{y_n}\left(1 + z_n\frac{\varphi(x_n)}{x_n}\right)\frac{x_n}{\varphi(x_n)}.$$

So, again the witness sequence is not divergent, which is a contradiction. □

This gives the following theorem.

THEOREM 4B (Beurling–Darboux UCT: Baire version). *If $\varphi \in \mathrm{BSV}$ has the Baire and Darboux properties, then $\varphi \in \mathrm{SN}$: (BSV) holds locally uniformly.*

Proof. Suppose not; take $h = \log\varphi$. Then there exists a witness sequence v_n with limit v, and in particular, for some $x_n \to \infty$ and $\varepsilon_0 > 0$, one has inequality (1) modified so that v_n replaces u_n.

We construct below a convergent sequence u_n, with limit u say, such that

$$c_n := h(x_n + u_n\varphi(x_n)) - h(x_n) \longrightarrow c \in (-\infty, \infty), \tag{6}$$

and also the unmodified (1) holds. This will contradict Theorem 2B.

The proof here splits according as $h(x_n + v_n\varphi(x_n)) - h(x_n)$ versus are bounded.

Case (i). The differences $h(x_n + v_n\varphi(x_n)) - h(x_n)$ diverge to $\pm\infty$. Here we appeal to the Darboux property to replace the sequence $\{v_n\}$ by another sequence $\{u_n\}$ for which the corresponding differences are convergent.

Now $f_n(t) = h(x_n + t\varphi(x_n)) - h(x_n)$ has the Darboux property and $f_n(0) = 0$. Either $f_n(v_n) \geqslant \varepsilon_0$, and so there exists u_n between 0 and v_n with $f_n(u_n) = \varepsilon_0$, or $-f_n(v_n) \geqslant \varepsilon_0$, and so there exists u_n with $-f_n(u_n) = \varepsilon_0$. Either way, $|f_n(u_n)| = \varepsilon_0$. Without loss of generality, $\{u_n\}$ is convergent with limit u, say, since $\{v_n\}$ is so, and now (6) and (1) hold, the latter as in fact

$$|h(x_n + u_n\varphi(x_n)) - h(x_n)| = \varepsilon_0.$$

Case (ii). $h(x_n + v_n\varphi(x_n)) - h(x_n)$ versus are bounded. In this case, we can get (2) by passing to a subsequence.

In either case, we contradict Theorem 2B. □

Appealing instead to Theorem 2M, the same argument gives the following theorem.

THEOREM 4M (Beurling–Darboux UCT, measure version). *If $\varphi \in \mathrm{BSV}$ is measurable and has the Darboux property, then $\varphi \in \mathrm{SN}$: (BSV) holds locally uniformly.*

REMARKS. (1) The Darboux property in Theorems 4 may be replaced with a weaker local property. It is enough to require that φ be *locally range-dense*, that is, that at each point t there is a bounded open neighbourhood I_t such that the range $\varphi[I_t]$ is dense in the interval $(\inf\varphi[I_t], \sup\varphi[I_t])$, or be in the class $\mathfrak{A}_0$ of [**26**, § 2]; cf. also [**27**].

(2) The proofs of Theorems 4B and 4M begin as Bloom's does, but only in the case (i) of the first step, and even then we appeal to the Darboux property rather than the much stronger assumption of continuity. Thereafter, we are able to use Theorem CET to base the rest of the proof on Baire's category theorem. This enables us to handle Theorems 4B and 4M together, by qualitative measure theory; in contrast, the proofs of Bloom's theorem in [**21**] and BGT § 2.11 use quantitative measure theory, see § 10.2.

(3) Recall that Darboux [31] (cf. [26]) has proved that any derivative function has the Darboux, or intermediate property.

(4) That the Darboux property is natural here will emerge in § 6 on topological dynamics.

Functions of Baire class 1. Recall that Baire class 1 functions, briefly, Baire-1 functions, are limits of sequences of continuous functions, and then the Baire hierarchy is defined by successive passages to the limit. See, for example, Bruckner and Leonard [28, § 2], and the extensive bibliography given there, [66]; cf. [43, § 24.B]. The union of the classes in the Baire hierarchy gives the Borel functions; see, for example, [56, Chapter XV]. Since Borel functions are (Lebesgue) measurable and Baire (have the Baire property),† the Baire-1 functions are both measurable and Baire (see, for example, [47, § 11]).

Lee, Tang and Zhao [49] define a concept of *weak separation* involving neighbourhood assignments. They show that, for real-valued functions on a Polish space, this is equivalent to being of Baire class 1. Their result is greatly generalized by Bouziad [25].

Darboux functions of Baire class 1. We recall that a Darboux function need be neither measurable nor (with the property of) Baire, hence the need to impose Darboux–Lebesgue or Darboux–Baire as double conditions in our results (though 'Darboux' here is barely restrictive, from context: see § 5).

While Darboux functions in general may be badly behaved, Darboux functions of Baire class 1 are more tractable; recall the Kuratowski–Sierpiński theorem of § 4. See, for example, [26, § 6; 28, § 5; 29, 34] for Darboux functions of Baire class 1, and Marcus [35, 36, 50] for literature and illuminating examples in the study of the Darboux property.

Part II. Beurling regular variation

5. *The setting*

We begin by setting the context of what we call *Beurling regular variation*, extending the Beurling slow variation of Part I. We use the infinite combinatorics of § 2 to establish (in § 7) a Beurling analogue of the UCT of Karamata theory. In § 6, we discuss the flow issues raised below; there flow rates, time measures and cocycles are introduced. Here we discuss the connection between the orbits of the relevant flows and the Darboux property that plays such a prominent role in Part I. Incidentally, this explains why the Darboux property is quite natural there. These ideas prepare the ground for a Beurling version of the UCT. We deduce a *Characterization Theorem* in § 8. Armed with these two theorems, we are able in § 9 to establish various *Representation Theorems* for Beurling regularly varying functions, but only after a review of Bloom's work on the representation of self-neglecting functions, from which we glean *Smooth Variation Theorems*. We close in § 10 by commenting on the place of Karamata theory, and of de Haan's theory of the Γ-class, relative to the new Beurling theory.

Below the reader should have clearly in mind two isometric topological groups: the real line under addition with (the Euclidean topology and) Haar measure Lebesgue measure dx, and the positive half-line under multiplication, with Haar measure dx/x, and metric $d_W(x, y) = |\log y - \log x|$. ('W for Weil', as this generates the underlying Weil topology of the Haar measure, for which see [38, § 62; 68].) As usual, we will move back and forth between these two as may be convenient, by using their natural isomorphism exp/log. Again as usual, we work additively in

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†In general, one needs to distinguish between Borel and Baire measurability (cf. [38, § 51]), but the two coincide for real analysis, our context here, see [43, 24.3].

proofs, and multiplicatively in applications; we use the convention

$$h := \log f, \quad k := \log g.$$

As in § 1, we need to use both addition and multiplication simultaneously, so *Aff* is natural here. Recall that on the line the *affine group* $x \to ux + v$ with $u > 0$ and v real has (right) Haar measure $u^{-1}\,du\ dv$ (or $(du/u)\,dv$, as above), see [**42**, IV, (15.29)]. This explains the presence of the two measure components, the 'du/u' and the 'dv', in (Γ_ρ) of Theorem 10′ (§ 9). Generalizations of Karamata's theory of regular variation (BGT; cf. [**45**]) rely on a group G acting on a space X in circumstances where one can interpret 'limits to infinity' $x \to \infty$ in the following expression:

$$g(t) := \lim_{x\to\infty} f(tx)/f(x) \quad \text{for } t \in G \text{ and } x \in X.$$

Here an early treatment is [**1**] followed by [**2**], but a full topological development dates from our recent papers (for which see [**17**, **18**], and for the particular role of group action [**55**, **58**]). Recall that a group action $A : G \times X \to X$ requires two properties:

(i) *identity*: $A(1_G, x) = x$ for all x, that is, $1_G = \mathrm{id}$ and
(ii) *associativity*: $A(gh, x) = A(g, A(h, x))$,

with the maps $x \to g(x) := A(g, x)$, also written gx, often being homeomorphisms. An action A defines an *A-flow* (also referred to as a G-flow), whose orbits are the sets $Ax = \{gx : g \in G\}$.

In fact, (i) follows from (ii) for *surjective* A (as $A(g, y) = A(1_G, A(g, y))$), so we will say that A is a *pre-action* if just (i) holds, and then continue to use the notation $g(x) := A(g, x)$; it is helpful here to think of the corresponding sets Ax as orbits of an *A-preflow*, using the language of flows and topological dynamics [**6**].

Below we relax the definition of regular variation so that it relies not so much on group action but on asymptotic 'cocycle action' associated with a group G. This will allow us to develop a theory of Beurling regular variation analogous to the Karamata theory, in which the regularly varying functions are those Baire or measurable f which, for some fixed self-neglecting φ, possess a *non-zero* limit function g (not identically zero modulo null/meagre sets) satisfying

$$f(x + t\varphi(x))/f(x) \longrightarrow g(t) \quad \text{as } x \longrightarrow \infty, \ \forall t \tag{BRV}$$

(so that $g(0) = 1$). Equivalently,

$$h(x + t\varphi(x)) - h(x) \longrightarrow k(t), \quad \text{as } x \longrightarrow \infty, \ \forall t \tag{BRV$_+$}$$

(so that $k(0) = 0$). This latter equivalence is non-trivial: it follows from Theorem 7 that if g is a non-zero function, then it is in fact positive. Specializing (BRV) to the sequential format

$$g(t) = \lim_{n\in\mathbb{N}} f(n + t\varphi(n))/f(n),$$

one sees that the limit function g is Baire/measurable if f is so. We refer to functions f satisfying (BRV) as (Beurling) *φ-regularly varying*.

This takes us beyond the classical development of such a theory restricted to the class Γ of monotonic functions f (BGT § 3.10; de Haan [**37**]). We prove in Theorem 6 a UCT for Baire/measurable functions f with non-zero limit g, not previously known, and in Theorem 7, the Characterization Theorem, that a Baire/measurable function f is Beurling φ-regularly varying with non-zero limit if and only if, for some ρ, one has

$$f(x + t\varphi(x))/f(x) \longrightarrow e^{\rho t} \quad \forall t,$$

where ρ is the *Beurling φ-index* of regular variation. Baire and measurable (positive) functions of this type form the class $\Gamma_\rho(\varphi)$ (cf. Omey [**57**], for f measurable; see also [**32**] for the analogous power-wise approach to Karamata regular variation).

6. *Topological dynamics*: *flows*, *orbits*, *cocycles*

Our approach is to view Beurling regular variation as a generalization of Karamata regular variation obtained by replacing the associativity of group action by a form of asymptotic associativity. To motivate our definition below, take X and G both to be $(\mathbb{R},+)$, $\varphi \in \mathrm{SN}$ and consider the map

$$T^{\varphi} : (t,x) \longmapsto x + t\varphi(x).$$

Think of t as representing translation. For fixed t put

$$t(x), \quad \text{or just } tx := T_t^{\varphi}(x) = T^{\varphi}(t,x) = x + t\varphi(x),$$

so that $0(x) = x$, and so T^{φ} is a pre-action. Here we have $T_{s+t}^{\varphi}(x) = x + (s+t)\varphi(x)$, so that

$$T_s^{\varphi}(T_t^{\varphi}(x)) = x + t\varphi(x) + s\varphi(x + t\varphi(x)) \neq T_{s+t}^{\varphi}(x).$$

So $T^{\varphi} : G \times X \to X$ is not a group action, as associativity fails. However, just as in a proper flow context, here too one has a well-defined *flow rate*, or infinitesimal generator, at x, for which see [**6**, **10**, **65**, 13.34] (cf. [**2**]),

$$\dot{T}_0^{\varphi} x = \frac{d}{dt} T_t^{\varphi} x \bigg|_{t=0} = \lim_{t \to 0} \frac{T_t^{\varphi} x - x}{t} = \varphi(x).$$

There is of course an underlying true flow here, in the measure case, generated[†] by $\varphi > 0$ (with $1/\varphi$ locally integrable) and described by the system of differential equations (writing $u_x(t)$ for $u(t,x)$)

$$\dot{u}_x(t) = du_x(t)/dt = \varphi(u_x(t)) \quad \text{with } u_x(0) = x, \text{ so that } (t,x) \longmapsto u_x(t). \tag{7}$$

(The inverse problem, for $t(u)$ with $t(0) = 1$, has an explicit increasing integral representation, yielding $u_x(t) := u(t + t(x))$, where $u(t(x)) = x$, as u and t are inverse.) The 'differential flow' $\Phi : (t,x) \mapsto u_x(t)$ is continuous in t for each x. As such, Φ is termed by Beck a *quasi-flow*.[‡] In contrast 'translation flow', that is, $(t,x) \mapsto x + t$, being jointly continuous, is a 'continuous flow', briefly a *flow*. It is interesting to note that, by a general result of Beck (see [**6**, Chapter 4, Reparametrization, Theorem 4.4]), if the *orbits* of Φ (that is, the sets $\mathcal{O}(x) := \{\Phi(t,x) : t \in \mathbb{R}\}$) are continua then, even though φ need not be continuous, there still exists a unique *continuous* 'local time-change' system of maps $t \mapsto f_x(t)$ embedding the quasi-flow in the translation flow, that is, $\Phi(t,x) = x + f_x(t)$; here f_x has the *cocycle property* (cf. Theorem 5),

$$f_x(s+t) = f_x(s) + f_y(t) \quad \text{for } y = x + f_x(s),$$

and $f_x(0) = 0$ for all x. This will be the case when φ has the intermediate value property, and so here the Darboux property says simply that *orbits embed*. Cocycles are thus central to the flow analysis of regular variation.

It is this differential flow that the, algebraically much simpler, Beurling preflow circumvents, working not with the continuous translation function $f_x(t)$ but $t\varphi(x)$, now only measurable, but with the variables separated. Nevertheless, the differential equation above is the source of an immediate interpretation of the integral

$$\tau_x := \int_1^x \frac{du}{\varphi(u)},$$

[†] Positivity is key here; $x = 0$ is a fixed point of the flow $\dot{u} = \varphi(u)$ when $\varphi(x) = \sqrt{|x|}$.

[‡] Beck denotes flows by $\varphi(t,x)$ and uses f where we use φ. As we follow the traditional notation of φ for self-neglecting functions, the flow here is denoted by Φ.

arising in the representation formula (Γ_ρ) of Theorem 10′ (§ 9) for a regularly varying function f, as the metric of *time measure* (in the sense of Beck [6, p. 153]). The metric is the *occupation-time measure* (cf. BGT § 8.11) of the interval $[1, x]$ under the φ-generated flow started at the natural origin of the multiplicative group $\mathbb{R}_+$. For $\varphi(x) = x$,[§] the φ-time measure is Haar measure, and the associated metric is the Weil (multiplicatively invariant) metric with $d_W(1, x) = |\log x|$, as in § 5. In general, however, the φ-time measure μ_φ is obtained from Haar measure via the density $x/\varphi(x)$, interpretable as a *time-change* 'multiplier' $w(x) := \varphi(x)/x$ (cf. [6, 5.41]).

Granted its interpretation, it is only to be expected in (Γ_ρ) that τ_x multiplies the index ρ describing the asymptotic behaviour of the function f. The time integral τ_x is in fact asymptotically equal to the time taken to reach x from the origin under the Beurling pre-action T^φ, when $\varphi \in \mathrm{SN}$, namely $x/\varphi(x)$. We hope to return to this matter elsewhere.

Actually, T^φ is even closer to being an action: it is an *asymptotic action* (that is, asymptotically an action), in view of two properties critical to the development of regular variation. The first (for the second see below Lemma 4) refers to the dual view of the map $(t, x) \mapsto x(t) = x + t\varphi(x)$ with x fixed (rather than t, as at the beginning of the section). Here we see the affine transformation $\alpha_x(t) = \varphi(x)t + x$. This *auxiliary group* plays its part through allowing the absorption of a small 'time' variation $t + s$ of t into a small 'space' variation in x. This involves a concatenation formula, earlier identified in [10] as a component in the abstract theory of the index of regular variation.

LEMMA 4 (Near-associativity, almost absorption). *For $\varphi \in \mathrm{SN}$, one has*

$$T^\varphi(t+s, x) = T^\varphi(\gamma t, T^\varphi(s, x)) \quad \textit{where } \gamma = \gamma_x^\varphi(y_s) = \varphi(x)/\varphi(y) \textit{ and } y_s := T^\varphi(s, x)$$

and

$$\gamma_x^\varphi(y_s) \longrightarrow 1 \quad \textit{as } s \longrightarrow 0.$$

Here γ satisfies the concatenation formula

$$\gamma_x^\varphi(z) = \gamma_x^\varphi(y)\gamma_y^\varphi(z) \quad \forall x, y, z.$$

Alternatively,

$$T_{t+s}^\varphi x = T_{\gamma t}^\varphi T_s^\varphi x = T_{\gamma t}^\varphi y_s, \quad \textit{where } y_s = \alpha_x(s) = s\varphi(x) + x = T_s^\varphi x,$$

equivalently

$$T_{t+s}^\varphi x = T_{\beta(t+s)}^\varphi \alpha_x(s) \quad \textit{for } \beta(t) = \gamma_x(y_s)(t - s).$$

Proof.

$$T_{t+s}^\varphi x = x + (t+s)\varphi(x) = (x + s\varphi(x)) + \frac{\varphi(x)}{\varphi(x + s\varphi(x))} t\varphi(x + s\varphi(x)).$$

As for the concatenation formula, one has

$$\gamma_x^\varphi(z) = \frac{\varphi(x)}{\varphi(z)} = \frac{\varphi(x)}{\varphi(y)} \cdot \frac{\varphi(y)}{\varphi(z)} = \gamma_x^\varphi(y)\gamma_y^\varphi(z). \qquad \square$$

As for the second property of T^φ, recall that, for G a group acting on a second group X, a G-cocycle on X is a function $\sigma : G \times X \to X$ defined by the condition

$$\sigma(gh, x) = \sigma(g, hx)\sigma(h, x).$$

[§] See § 10.3, where the restriction $\varphi(x) = o(x)$ is dropped.

This definition is already meaningful if a pre-action rather than an action is defined from $G \times X$ to X; and so for the purposes of asymptotic analysis one may capture a weak form of associativity as follows using Lemma 4. For a Banach algebra X, we let X^{-1} denote the set of its invertible elements.

DEFINITION. For X a Banach algebra, given a pre-action $T : G \times X \to X$ (that is, with $1_G x = x$ for all x, where, as above, $gx := T(g, x)$), an *asymptotic G-cocycle* on X is a map $\sigma : G \times X \to X^{-1}$ with the property that, for all $g, h \in G$ and $\varepsilon > 0$, there is $r = r(\varepsilon, g, h)$ such that, for all x with $\|x\| > r$,

$$\|\sigma(gh, x) - \sigma(g, hx)\sigma(h, x)\|_X < \varepsilon.$$

Say that the cocycle is *locally uniform* if the inequality holds uniformly on compact (g, h)-sets.

REMARK. Baire and measurable cocycles are studied in [**58**] for their uniform boundedness properties. One sees that the Second and Third Boundedness Theorems proved there hold in the current setting with asymptotic cocycles replacing cocycles. We now verify that replacing X^{-1} by $\mathbb{R}_+$, and taking $T = T^\varphi$ and the natural cocycle of regular variation $\sigma^f(t, x) := f(tx)/f(x)$, the above property holds. The case $f = \varphi$ comes first; the general case of φ-regularly varying f must wait till after Theorem 7.

THEOREM 5. *For (positive) $\varphi \in \mathrm{SN}$, the map $\sigma^\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$,*

$$\sigma^\varphi(t, x) := \varphi(tx)/\varphi(x) \quad \textit{where } tx := T_t^\varphi(x),$$

regarded as a map into the Banach algebra $\mathbb{R}$, is a locally uniform asymptotic $(\mathbb{R}, +)$-cocycle, that is, for every $\varepsilon > 0$ and compact set K, there is r such that, for all $s, t \in K$ and all x with $\|x\| > r$,

$$|\sigma^\varphi(s + t, x) - \sigma^\varphi(s, tx)\sigma^\varphi(t, x)| < \varepsilon,$$

that is,

$$\left| \frac{\varphi(T_{s+t}^\varphi(x))}{\varphi(x)} - \frac{\varphi(T_s^\varphi(T_t^\varphi(x)))}{\varphi(T_t^\varphi(x))} \cdot \frac{\varphi(T_t^\varphi(x))}{\varphi(x)} \right| < \varepsilon$$

or

$$|\varphi(T_{s+t}^\varphi(x))/\varphi(x) - \varphi(T_s^\varphi(T_t^\varphi(x)))/\varphi(x)| < \varepsilon.$$

Proof. Let $\varepsilon > 0$. Given s, t, let I be any open interval with $s + t \in I$.

Pick $\delta > 0$ so that the interval $J = (1 - \delta, 1 + \delta)$ satisfies $t + sJ \subseteq I$. Next pick r such that, for $x > r$, both

$$|\sigma^\varphi(t, x) - 1| = |\varphi(x + t\varphi(x))/\varphi(x) - 1| < \delta$$

and

$$|\sigma^\varphi(u, x) - 1| = |\varphi(x + u\varphi(x))/\varphi(x) - 1| < \varepsilon/2 \quad \text{for all } u \in I.$$

In particular,

$$|\sigma^\varphi(s + t, x) - 1| = |\varphi(x + (s + t)\varphi(x))/\varphi(x) - 1| < \varepsilon/2.$$

Noting, as in Lemma 4, that

$$T_s^\varphi(T_t^\varphi x) = (x + t\varphi(x)) + s\varphi(x + t\varphi(x)) = x + \varphi(x)\left(t + s\frac{\varphi(x + t\varphi(x))}{\varphi(x)}\right),$$

so that

$$w := t + s\sigma^{\varphi}(t,x) = t + s\frac{\varphi(x+t\varphi(x))}{\varphi(x)} \in t + sJ \subseteq I,$$

one has

$$|\sigma^{\varphi}(w,x) - 1| < \varepsilon/2 : \ |\varphi(T_s^{\varphi}(T_t^{\varphi}x))/\varphi(x) - 1| < \varepsilon/2.$$

But

$$\sigma^{\varphi}(w,x) = \frac{\varphi(T_s^{\varphi}(T_t^{\varphi}x))}{\varphi(x)} = \frac{\varphi(T_s^{\varphi}(T_t^{\varphi}(x)))}{\varphi(T_t^{\varphi}(x))}\frac{\varphi(T_t^{\varphi}(x))}{\varphi(x)} = \sigma^{\varphi}(s, T_t^{\varphi}(x))\sigma^{\varphi}(t,x),$$

so, for $x > r$, one has

$$\begin{aligned} |\sigma^{\varphi}(w,x) - \sigma^{\varphi}(s+t,x)| &\leqslant |[\sigma^{\varphi}(s,T_t^{\varphi}(x))\sigma^{\varphi}(t,x)] - 1| + |\sigma^{\varphi}(s+t,x)] - 1| \\ &\leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

□

7. *Uniform convergence theorem*

We begin with a lemma that yields simplifications later. As before, we call it a Shift Lemma. It has substantially the same statement concerning local uniformity and proof as Lemma 3 § 4 except that here $h = \log f$, whereas there one has $h = \log\varphi$, so that here the difference $h(x_n + u\varphi(x_n)) - h(x_n)$ tends to $k(u)$ rather than to zero. So we omit the proof.

Lemma 5 (Shift Lemma: uniformity preservation under shift). *For any u, convergence in* (BRV$_+$) *is uniform near $t = 0$ if and only if it is uniform near $t = u$.*

Definition. Say that $\{u_n\}$ with limit u is a *witness sequence at u* (for non-uniformity in h) if there are $\varepsilon_0 > 0$ and a divergent sequence x_n such that, for $h = \log f$,

$$|h(x_n + u_n\varphi(x_n)) - h(x_n)| > \varepsilon_0 \quad \forall n \in \mathbb{N}. \tag{8}$$

Theorem 6 (UCT for φ-regular variation). *For $\varphi \in$ SN, if f has the Baire property* (*or is measurable*) *and satisfies* (BRV) *with limit g strictly positive on a non-negligible set, then f is locally uniformly φ-RV.*

Proof. Suppose otherwise. We modify a related proof from § 4 (concerned there with the special case of φ itself) in two significant details. In the first place, we will need to work relative to the set $S := \{s > 0 : g(s) > 0\}$ ('S' for support), so that $k(s) = \log g(s)$ is well defined on S. Now S is Baire/measurable; as S is non-negligible, by passing to a Baire/measurable subset of S if necessary, we may assume without loss of generality that the restriction $k|S$ is continuous on S, by [**47**, § 28] in the Baire case and Luzin's Theorem in the measure case [**60**, Chapter 8]; [**43**, 17.12]).

Let u_n be a witness sequence for the non-uniformity of h and so, for some $x_n \to \infty$ and $\varepsilon_0 > 0$, one has (8). By the Shift Lemma (Lemma 5),we may assume that $u = 0$. So we will write z_n for u_n. As φ is self-neglecting,

$$c_n := \varphi(x_n + z_n\varphi(x_n))/\varphi(x_n) \longrightarrow 1. \tag{9}$$

Write $\gamma_n(s) := c_n s + z_n$ and $y_n := x_n + z_n\varphi(x_n)$. Then $y_n = x_n(1 + z_n\varphi(x_n)/x_n) \to \infty$, (as $\varphi(x) = o(x)$, see Theorem 8) and, as $k(0) = 0$,

$$|h(y_n) - h(x_n)| \geqslant \varepsilon_0. \tag{10}$$

Now take $\eta = \varepsilon_0/4$ and, for $x = \{x_n\}$, working in S, put

$$V_n^x(\eta) := \{s \in S : |h(x_n + s\varphi(x_n)) - h(x_n) - k(s)| \leqslant \eta\}, \quad H_k^x(\eta) := \bigcap_{n \geqslant k} V_n^x(\eta),$$

and likewise for $y = \{y_n\}$. These are Baire sets, and

$$S = \bigcup_k H_k^x(\eta) = \bigcup_k H_k^y(\eta), \tag{11}$$

as $h \in \mathrm{BRV}_+$. The increasing sequence of sets $\{H_k^x(\eta)\}$ covers S. So, for some k, the set $H_k^x(\eta)$ is non-negligible. As $H_k^x(\eta)$ is non-negligible, by (11), for some l the set

$$B := H_k^x(\eta) \cap H_l^y(\eta)$$

is also non-negligible. Taking $A := H_k^x(\eta)$, one has $B \subseteq H_l^y(\eta)$ and $B \subseteq A$ with A, B non-negligible. Applying Lemma 2 to the maps $\gamma_n(s) = c_n s + z_n$ with $c = \lim_n c_n = 1$, there exist $b \in B$ and an infinite set $\mathbb{M}$ such that

$$\{c_m b + z_m : m \in \mathbb{M}\} \subseteq A = H_k^x(\eta).$$

That is, as $B \subseteq H_l^y(\eta)$, there exist $t \in H_l^y(\eta)$ and an infinite $\mathbb{M}_t$ such that

$$\{\gamma_m(t) = c_m t + z_m : m \in \mathbb{M}_t\} \subseteq H_k^x(\eta).$$

In particular, for this t and $m \in \mathbb{M}_t$ with $m > k, l$ one has

$$t \in V_m^y(\eta) \text{ and } \gamma_m(t) \in V_m^x(\eta).$$

As $t \in S$ and $\gamma_m(t) \in S$ (a second critical detail), we have by the continuity of $k|S$ at t, since $\gamma_m(t) \to t$, that, for all m large enough

$$|k(t) - k(\gamma_m(t))| \leqslant \eta. \tag{12}$$

Fix such an m. As $\gamma_m(t) \in V_m^x(\eta)$,

$$|h(x_m + \gamma_m(t)\varphi(x_m)) - h(x_m) - k(\gamma_m(t))| \leqslant \eta. \tag{13}$$

But $\gamma_m(t) = c_m t + z_m = z_m + t\varphi(y_m)/\varphi(x_m)$, so

$$x_m + \gamma_m(t)\varphi(x_m) = x_m + z_m\varphi(x_m) + t\varphi(y_m) = y_m + t\varphi(y_m),$$

'absorbing' the affine shift $\gamma_m(t)$ into y. So, by (13),

$$|h(y_m + t\varphi(y_m)) - h(x_m) - k(\gamma_m(t))| \leqslant \eta.$$

But $t \in V_m^y(\eta)$, so

$$|h(y_m + t\varphi(y_m)) - h(y_m) - k(t)| \leqslant \eta.$$

Combining these with (12),

$$\begin{aligned} |h(y_m) - h(x_m)| &\leqslant |h(y_m + t\varphi(y_m)) - h(y_m) - k(t)| + |k(t) - k(\gamma_m(t))| \\ &\quad + |h(y_m + t\varphi(y_m)) - h(x_m) - k(\gamma_m(t))| \\ &\leqslant 3\eta < \varepsilon_0, \end{aligned}$$

contradicting (10). □

We have seen in the Beurling UCT how uniformity in the auxiliary function φ passes 'out' to φ-regularly varying f. For the converse (uniformity passing 'in' from f to φ), we note the following.

PROPOSITION. *For Baire/measurable f, if, for some Baire/measurable function $\varphi > 0$ and some real $\rho \neq 0$,*

$$f(x + u\varphi(x))/f(x) \longrightarrow e^{\rho u}, \quad \textit{locally uniformly in } u,$$

then $\varphi \in \mathrm{SN}$.

Proof. Replace u by ρu and $\varphi(x)$ by $\psi(x) = \varphi(x)/\rho$ to yield

$$f(x + u\psi(x))/f(x) \longrightarrow e^{u}$$

(allowing $u < 0$); then follow verbatim as in BGT, 3.10.6, which relies only on uniformity (and on the condition that $x + u\psi(x) \to \infty$ as $x \to \infty$, which is also deduced from uniformity in BGT 3.10.1).

8. *Characterization theorem*

We may now deduce the characterization theorem, which implies in particular that the support set S of the proof of Theorem 6 is in fact all of $\mathbb{R}$.

THEOREM 7 (Characterization Theorem). *For $\varphi \in \mathrm{SN}$, if $f > 0$ is φ-regularly varying and Baire/measurable and satisfies*

$$f(x + t\varphi(x))/f(x) \longrightarrow g(t) \quad \forall t,$$

with non-zero limit, that is, $g > 0$ on a non-negligible set, then, for some ρ (the index of φ-regular variation), one has

$$g(t) \equiv e^{\rho t}.$$

Proof. Proceed as in Theorem 5: writing $y := x + s\varphi(x)$, and recalling from Theorem 1 the notation $\gamma = \varphi(x)/\varphi(y)$, one has

$$h(x + (s + t)\varphi(x)) - h(x) = [h(y + t\gamma\varphi(y)) - h(y)] + [h(y) - h(x)]. \tag{14}$$

Fix s and $t \in \mathbb{R}$; passing to limits and using uniformity (by Theorem 6),

$$k(s + t) = k(t) + k(s), \tag{CFE}$$

since $\gamma = \varphi(x)/\varphi(y) \to 1$. This is the Cauchy functional equation; as is well known, for k Baire/measurable (see Banach [**5**, Chapter I, § 3, Theorem 4] and Mehdi [**51**] for the Baire case, [**46**, 9.4.2] for the measure case, and [**15**] for an up-to-date discussion) this implies $k(x) = \rho x$ for some $\rho \in \mathbb{R}$, and so $g(x) = e^{\rho x}$. □

REMARK. The conclusion that $k(x) = \rho x$ $(\forall x)$ for some ρ tells us that in fact $g > 0$ everywhere, which in turn implies the cocycle property below. (If we assumed that $g > 0$ everywhere, then we could argue more directly, and more nearly as in the Karamata theory, by establishing the cocycle property first and from it deducing the Characterization Theorem.)

As an immediate corollary, we now have an extension to Theorem 5.

COROLLARY 1 (Cocycle property). *For $\varphi \in \mathrm{SN}$, if $f > 0$ is φ-regularly varying and Baire/measurable and satisfies*

$$f(x + t\varphi(x))/f(x) \longrightarrow g(t) \quad \forall t,$$

with non-zero limit, that is, $g > 0$ on a non-negligible set, then

$$\sigma^f(t,x) := f(x + t\varphi(x))/f(x)$$

is a locally uniform asymptotic cocycle.

Proof. With the notation of Theorem 7, rewrite (14) as

$$\frac{f(x+(s+t)\varphi(x))}{f(x)} = \frac{f(y+t\gamma\varphi(y))}{f(y)} \cdot \frac{f(x+s\varphi(x))}{f(x)}, \tag{15}$$

where by Theorem 7 both ratios on the right-hand side have non-zero limits $g(t)$ and $g(s)$, as x (and so y) tend to infinity. Given $\varepsilon > 0$, it now follows from (15) using (CFE), and uniformity in compact neighbourhoods of s, t and $s+t$ (by Theorem 6), that, for all large enough x,

$$\left| \frac{f(x+(s+t)\varphi(x))}{f(x)} - \frac{f(x+t\varphi(x))}{f(x)} \cdot \frac{f(x+s\varphi(x))}{f(x)} \right| < \varepsilon,$$

so that σ^f is a locally uniform asymptotic cocycle. □

9. *Smooth variation and representation theorems*

Before we derive a representation theorem for Baire/measurable Beurling regularly varying functions, we need to link the Baire case to the measure case. Recall the *Beck iteration* of $\gamma(x) := T_1^{\varphi}(x) = x + \varphi(x)$ (so that $\gamma_{n+1}(x) = \gamma(\gamma_n(x))$ with $\gamma_1 = \gamma$, for which see [**6**, 1.64] in the context of bounding a flow) and Bloom's result for $\varphi \in \mathrm{SN}$ concerning the sequence $x_{n+1} = \gamma_n(x_1)$, that is, $x_{n+1} := x_n + \varphi(x_n)$, that, for all x_1 large enough, one has $x_n \to \infty$, that is, the sequence gives a *Bloom partition* of $\mathbb{R}_+$ (see [**21**], or BGT § 2.11). We next need to recall a construction due to Bloom in detail as we need a slight amendment.

Lemma 7 (Interpolation Lemma). *For $\varphi \in \mathrm{SN}$, set $x_{n+1} := x_n + \varphi(x_n)$, with x_1 large enough so that $x_n \to \infty$. Put $x_0 = 0$.*

For $\psi > 0$ a φ-slowly varying function, there exists a continuously differentiable function $\phi > 0$ such that

(i) *$\phi(x_n) = \psi(x_n)$ for $n = 0, 1, 2, \ldots$;*
(ii) *$\phi(x)$ lies between $\psi(x_n)$ and $\psi(x_{n-1})$ for x between x_{n-1} and x_n for $n = 1, 2, \ldots$;*
(iii) *$|\phi'(x)| \leqslant 2|\psi(x_n) - \psi(x_{n-1})|/\varphi(x_{n-1})$ with x between x_{n-1} and x_n for $n = 1, 2, \ldots$.*

Proof. Proceed as in [**21**] or BGT § 2.11; we omit the details. □

Definition. Call any function ϕ with the properties (i)–(iii) a (φ-) *interpolating function* for ψ.

We now deduce an extension of the Bloom–Shea Representation Theorem in the form of a Smooth Beurling Variation Theorem (for smooth variation, see BGT § 2.1.9, following Balkema *et al.* [**4**]). Indeed, the special case $\psi = \varphi$ is included here. Our proof is a variant on Bloom's. It will be convenient to introduce the following definition.

Definition (Asymptotic equivalence). For $\varphi, \phi > 0$, write $\varphi \sim \phi$ if $\varphi(x)/\phi(x) \to 1$. If $\phi \in \mathcal{C}^1$, then say that ϕ is a *smooth representation* of φ.

Theorem 8 (Smooth Beurling Variation). *For $\varphi \in \mathrm{SN}$ and ψ a φ-slowly varying function, if ϕ is any continuously differentiable function interpolating ψ with respect to φ as in Lemma 7, then*

$$\psi(x) = c(x)\phi(x) \quad \text{for some positive } c(\cdot) \to 1, \text{ that is, } \psi \sim \phi \in \mathcal{C}^1,$$

so that ϕ is φ-slowly varying, and also

$$\left|\varphi(x)\frac{\phi'(x)}{\phi(x)}\right| \longrightarrow 0.$$

This yields the representation

$$\psi(x) = c(x)\phi(x) = c(x)\exp\left(\int_0^x \frac{e(u)}{\varphi(u)}\,du\right) \quad \text{for } e \in \mathcal{C}^1 \text{ with } e \longrightarrow 0.$$

Moreover, if ψ is Baire/measurable, then so is $c(x)$.

Furthermore, if $\psi \in \mathrm{SN}$, in particular when $\psi = \varphi$, then:

(i) *$\phi(x)$ is self-neglecting and $\psi \sim \phi \sim \int_0^x e(u)\,du$ for some continuous e with $e \to 0$;*
(ii) *both $\phi(x)/x \to 0$ and $\psi(x)/x \to 0$, as $x \to \infty$;*
(iii) *if f is ψ-regularly varying with index ρ, then f is ϕ-regularly varying with index ρ with $\psi \sim \phi \in \mathcal{C}^1 \cap \mathrm{SN}$.*

Proof. Note that, for any y_n between x_n and x_{n+1}, one has $\varphi(y_n)/\varphi(x_n) \to 1$, and so also $\psi(y_n)/\psi(x_n) = [\psi(y_n)/\varphi(x_n)][\varphi(x_n)/\varphi(y_n)][\varphi(y_n)/\psi(x_n)] \to 1$; indeed $y_n = x_n + t_n\varphi(x_n)$ for some $t_n \in [0,1]$, and so the result follows from local uniformity in ψ and φ and because ψ is φ-slowly varying. This implies first that if (say) $\psi(x_n) \leqslant \psi(x_{n+1})$, then, for ϕ as in the theorem,

$$\frac{\psi(x_n)}{\psi(x_{n+1})} \leqslant \frac{\phi(y_n)}{\psi(y_n)} \leqslant \frac{\psi(x_{n+1})}{\psi(y_{n+1})},$$

and so $\phi(y_n)/\psi(y_n) \to 1$; similarly for $\psi(x_{n+1}) \leqslant \psi(x_n)$. So $\phi(x)/\psi(x) \to 1$, as $x \to \infty$. So, by Lemma 7(i) and as ψ is φ-slowly varying,

$$\frac{\phi(y_n)}{\varphi(x_n)} = \frac{\phi(y_n)}{\psi(y_n)} \cdot \frac{\psi(y_n)}{\varphi(x_n)} = \frac{\psi(y_n)}{\psi(y_n)} \cdot \frac{\psi(y_n)}{\varphi(x_n)} \longrightarrow 1,$$

that is, ϕ is φ-slowly varying. Furthermore, by Lemma 7(iii) and since ϕ and ψ are φ-slowly varying,

$$\left|\frac{\varphi(x)}{\phi(x)}\phi'(x)\right| \leqslant 2\left(\frac{\psi(x_{n+1})}{\psi(x_n)} - 1\right)(\psi(x_n)/\varphi(x_n)) \cdot (\varphi(x)/\phi(x)) \longrightarrow 0.$$

Take $c(x) := \psi(x)/\phi(x)$; then $\lim_{x\to\infty} c(x) = 1$, and rearranging one has

$$\psi(x) = c(x)\phi(x).$$

From here we have, setting $e(u) := \varphi(u)\phi'(u)/\phi(u) \to 0$ and noting that $e(x)/\varphi(x) \geqslant 0$ is the derivative of $\log\phi(x)$,

$$\psi(x) = c(x)\phi(x) = c(x)\exp\int_0^x \frac{e(u)}{\varphi(u)}du.$$

Conversely, such a representation yields slow φ-variation: by the Mean Value Theorem, for any t there is $s = s(x) \in [0,t]$ such that

$$\int_x^{x+t\varphi(x)} \frac{e(u)}{\varphi(u)}\,du = \frac{e(x+s\varphi(x))}{\varphi(x+s\varphi(x))}t\varphi(x),$$

which tends to 0 uniformly in t as $x \to \infty$, since $e(\cdot) \to 0$ and $\varphi \in \mathrm{SN}$.

Now suppose additionally that $\psi \in \mathrm{SN}$ (for example, if $\psi = \varphi$). We check that then $\phi \in \mathrm{SN}$. Indeed, suppose that $u_n \to u$; then as $\phi(x_n) = \psi(x_n)$, and since $\psi \in \mathrm{SN}$, writing $y_n = x_n + u_n\phi(x_n)$, one has

$$\frac{\phi(y_n)}{\phi(x_n)} = \frac{\psi(x_n + u_n\phi(x_n))/c(y_n)}{\psi(x_n)/c(x_n)} = \frac{\psi(x_n + u_n\psi(x_n))/c(y_n)}{\psi(x_n)/c(x_n)} \longrightarrow 1,$$

as asserted in (i). Given this ϕ apply Lemma 7 with ϕ for φ and $\psi = \varphi = \phi$ to yield a further smooth representing function $\bar{\phi} \sim \phi$. Then, by Lemma 7(iii), we have, for a corresponding sequence $\bar{x}_n$,

$$|\bar{\phi}'(x)| \leqslant 2|\phi(\bar{x}_n) - \phi(\bar{x}_{n-1})|/\phi(\bar{x}_{n-1}) = 2|\phi(\bar{x}_n)/\phi(\bar{x}_{n-1}) - 1| \longrightarrow 0.$$

So, taking $e(x) = \bar{\phi}'(x)$, one has $\lim_{x\to\infty} e(x) = 0$ and, for $\bar{c}(x) := \phi(x)/\bar{\phi}(x)$, one has again $\lim_{x\to\infty} \bar{c}(x) = 1$. So integrating $\bar{\phi}'$, one has

$$\phi(x) = \bar{c}(x)\bar{\phi}(x) = \bar{c}(x)\int_0^x e(u)\,du \quad \text{with } e(u) \longrightarrow 0.$$

From this integral representation, ϕ is self-neglecting (as in [21], BGT § 2.11).

As to (ii), we first prove this for φ itself . So, specializing (i) to $\psi = \varphi$,

$$\varphi(x) \sim \int_0^x e(u)\,du \quad \text{with } e(u) \longrightarrow 0,$$

so $\varphi(x)/x \sim \int_0^x e(u)\,du/x \to 0$.

We now use the fact that $\varphi(x)/x \to 0$ to consider a general $\psi \in \mathrm{SN}$ that is φ-slowly varying. Take $\psi \sim \phi \in \mathcal{C}^1$. Take $a_n = \psi(x_n)/\varphi(x_n)$, $b_n = \varphi(x_n)/\varphi(x_{n-1}) > 0$, so that $a_n \to 1$ and $b_n \to 1$. Put $z_n := \varphi(x_n)/x_n > 0$, so that $z_n \to 0$, as just shown. Now one has by Lemma 7(i) that

$$\begin{aligned}\frac{\phi(x_n)}{x_n} &= \frac{\psi(x_n)}{x_{n-1} + \varphi(x_n)} = \frac{a_n}{1 + x_{n-1}/\varphi(x_n)}\\ &= \frac{a_n}{1 + (1/z_{n-1})\varphi(x_{n-1})/\varphi(x_n)} = \frac{a_n}{1 + 1/(z_{n-1}b_n)} \longrightarrow 0.\end{aligned}$$

As to (iii) for $\psi \in \mathrm{SN}$, if $\psi \sim \phi \in \mathcal{C}^1$, then $\phi \in \mathrm{SN}$ (by (ii)). So, as $\psi(x)/\phi(x) \to 1$, by Theorem 6

$$\lim_{x\to\infty} \frac{f(x + t\psi(x))}{f(x)} = \lim_{x\to\infty} \frac{f(x + t[\psi(x)/\phi(x)]\phi(x))}{f(x)}.$$

That is, f is ϕ-regularly varying. □

We have just seen that self-neglecting functions are necessarily $o(x)$. We now see that, for $\varphi \in \mathrm{SN}$, a φ-slowly varying function is also SN if it is $o(x)$.

THEOREM 9. *For $\varphi \in \mathrm{SN}$, if $\psi > 0$ is φ-slowly varying and $\psi(x) = o(x)$, then ψ is SN, and so has a representation*

$$\psi \sim \int_0^x e(u)\,du \quad \textit{with continuous } e(\cdot) \longrightarrow 0.$$

Proof. Since self-neglect is preserved under asymptotic equivalence, without loss of generality we may assume that ψ is smooth. Now $\psi(x)/\varphi(x) \to 1$ (by definition), and so, for fixed

u, $u[\psi(x)/\varphi(x)] \to u$. For ψ a φ-slowly varying function, by the UCT for φ-regular variation

$$\psi(x + t\varphi(x))/\varphi(x) \longrightarrow 1, \text{ locally uniform in } u,$$

as ψ is continuous. So, in particular,

$$\frac{\psi(x + t\psi(x))}{\psi(x)} = \frac{\psi(x + t[\psi(x)/\varphi(x)]\varphi(x))}{\varphi(x)} \frac{\varphi(x)}{\psi(x)} \longrightarrow 1.$$

That is, ψ is BSV, since $\psi(x) = o(x)$. But ψ is continuous, and so, by Bloom's theorem [**21**], $\psi \in \mathrm{SN}$. □

Lemma 8. *For measurable $\varphi \in \mathrm{SN}$, the function*

$$f_\rho(x) := \exp\left(\rho \int_1^x \frac{du}{\varphi(u)}\right)$$

is φ-regularly varying with index ρ.

Proof. With $h_\rho = \log f_\rho$, one has that

$$\begin{aligned}
&h_\rho(x + t\varphi(x)) - h_\rho(x) - \rho t \\
&\quad = \rho \int_x^{x+t\varphi(x)} \frac{du}{\varphi(u)} - \rho t = \rho \int_x^{x+t\varphi(x)} \left(\frac{\varphi(x)}{\varphi(u)} - 1\right) \frac{du}{\varphi(x)} \\
&\quad = \rho \int_0^t \left(\frac{\varphi(x)}{\varphi(x + v\varphi(x))} - 1\right) dv = o(1).
\end{aligned}$$

□

We may now establish our main result with f_ρ as above.

Theorem 10 (Beurling Representation Theorem). *For $\varphi \in \mathrm{SN}$ with φ Baire/measurable eventually bounded away from 0, and f measurable and φ-regularly varying: for some $\rho \in \mathbb{R}$ and φ-slowly varying function $\tilde{f}$, one has*

$$f(x) = f_\rho(x)\tilde{f}(x) = \exp\left(\rho \int_1^x \frac{du}{\varphi(u)}\right) \tilde{f}(x).$$

Any function of this form is φ-regularly varying with index ρ.

So $f \sim f_\rho \phi$ for some smooth representation ϕ of $\tilde{f}$.

Proof. By Theorem 8(iii), we may assume that φ is smooth. Choose ρ as in Theorem 7 and, referring to the flow rate $\varphi(x) > 0$ at x, put

$$\tilde{h}(x) := h(x) - \rho \int_1^x \frac{du}{\varphi(u)},$$

where $h = \log f$. So $\tilde{h}(x)$ is Baire/measurable as h is.

By Theorem 6 (UCT), locally uniformly in t one has a 'reduction' formula for $\tilde{h}$:

$$\tilde{h}(x + t\varphi(x)) - \tilde{h}(x) - \rho t + \rho \int_x^{x+t\varphi(x)} \frac{du}{\varphi(u)} = h(x + t\varphi(x)) - h(x) - \rho t = o(1).$$

So, substituting $u = x + v\varphi(x)$ in the last step,

$$\begin{aligned}\tilde{h}(x + t\varphi(x)) - \tilde{h}(x) &= \rho t - \rho \int_x^{x+t\varphi(x)} \frac{du}{\varphi(u)} + o(1)\\ &= \rho \int_x^{x+t\varphi(x)} \left(1 - \frac{\varphi(x)}{\varphi(u)}\right) \frac{du}{\varphi(x)} + o(1)\\ &= \rho \int_0^t \left(1 - \frac{\varphi(x)}{\varphi(x + v\varphi(x))}\right) dv + o(1)\\ &= o(1),\end{aligned}$$

and the convergence under the integral here is locally uniform in t since $\varphi \in \mathrm{SN}$. So $\exp(\tilde{h})$ is Beurling φ-slowly varying. The converse was established in Lemma 8. The remaining assertion follows from Theorem 8. □

As a second corollary of Theorems 6 and 7 and of the de Bruijn–Karamata Representation Theorem (see BGT, Theorems 1.3.1 and 1.3.3), we deduce a Representation Theorem for Beurling regular variation which extends previous results concerned with the class Γ, see BGT, Theorem 3.10.6. We need the following result, which is similar to Bloom's Theorem 4 except that we use regularity of φ rather than assume conditions on convergence rates.

LEMMA 9 (Karamata slow variation). *If $\varphi \in \mathrm{SN}$ with φ Baire/measurable eventually bounded away from 0, then*

$$\varphi(x + v)/\varphi(x) \longrightarrow 1 \quad \textit{as } x \longrightarrow \infty, \textit{ locally uniformly in } v.$$

Proof. Without loss of generality, suppose that $0 < K < \varphi(x)$ for all x. Fix v; then $0 \leqslant |v|/\varphi(x) \leqslant |v|K^{-1}$ for all x. Let $\varepsilon > 0$. Since $\varphi \in \mathrm{SN}$, there is $X = X(\varepsilon, v)$ such that

$$|\varphi(x + t\varphi(x))/\varphi(x) - 1| < \varepsilon, \tag{16}$$

for all $|t| \leqslant |v|K^{-1}$ and all $x \geqslant X$. So, in particular, for $x \geqslant X$ and $t := v/\varphi(x)$, since $|t| \leqslant |v|K^{-1}$, substitution in (16) yields

$$|\varphi(x + v)/\varphi(x) - 1| < \varepsilon,$$

for $x \geqslant X$. This shows that, for each $v \in \mathbb{R}$,

$$\varphi(x + v)/\varphi(x) \longrightarrow 1.$$

So $\log \varphi$ is Karamata slowly varying in the additive sense; being Baire/measurable, by the UCT of additive Karamata theory, convergence to the limit for $\log \varphi$, and so convergence for φ as above, is locally uniform in v. □

An alternative 'representation' follows from Lemma 9.

THEOREM 10′ (Beurling Representation Theorem). *For $\varphi \in \mathrm{SN}$ with φ Baire/measurable eventually bounded away from 0, and f measurable and φ-regularly varying: there are $\rho \in \mathbb{R}$, measurable $d(\cdot) \to d \in (0, \infty)$ and continuous $e(\cdot) \to 0$ such that*

$$\begin{aligned}f(x) &= d(x) \exp\left(\rho \int_1^x \frac{du}{\varphi(u)} + \int_0^x e(v)\, dv\right)\\ &= d(x) \exp\left(\rho \int_1^x \frac{u}{\varphi(u)} \frac{du}{u}\right) \exp\left(\int_0^x \frac{e(v)}{\varphi(v)}\, dv\right),\end{aligned} \tag{Γ_ρ}$$

where, for each t,

$$\int_x^{x+t\varphi(x)} e(v)\,dv = o(1).$$

Proof. Choose ρ as in Theorem 7. Referring to the flow rate $\varphi(x) > 0$ at x, put

$$\tilde{h}(x) := h(x) - \rho \int_1^x \frac{du}{\varphi(u)},$$

where $h = \log f$. Here, since $1/\varphi(x)$ is eventually bounded above as $x \to \infty$ and our analysis is asymptotic, without loss of generality we may assume again by Luzin's Theorem that φ here is continuous.

By Theorem 6 (UCT), locally uniformly in t one has, as in Theorem 10, a 'reduction' formula for $\tilde{h}$:

$$\begin{aligned}\tilde{h}(x+t\varphi(x)) - \tilde{h}(x) - \rho t + \rho \int_x^{x+t\varphi(x)} \frac{du}{\varphi(u)} \\ = h(x+t\varphi(x)) - h(x) - \rho t = o(1).\end{aligned} \tag{17}$$

Fix y and let $K > 0$ be a bound for $1/\varphi$, far enough to the right. We will use local uniformity in (17) on the interval $|t| \leqslant |y|K^{-1}$. First, take $t = y/\varphi(x)$, and so $|t| \leqslant |y|/K$, so by (17),

$$\begin{aligned}\tilde{h}(x+y) - \tilde{h}(x) &= \frac{\rho y}{\varphi(x)} - \rho \int_x^{x+y} \frac{du}{\varphi(u)} + o(1) \\ &= \rho \int_x^{x+y} \left(\frac{1}{\varphi(x)} - \frac{1}{\varphi(u)} \right) du + o(1) \\ &= \frac{\rho}{\varphi(x)} \int_0^y \left(1 - \frac{\varphi(x)}{\varphi(x+w)} \right) dw + o(1).\end{aligned}$$

By Lemma 9, applied to the set $\{v : |v| \leqslant |y|\}$, which corresponds to the w range in the integral above, we have

$$\tilde{h}(x+y) - \tilde{h}(x) = \frac{\rho}{\varphi(x)} \int_0^y \left(1 - \frac{\varphi(x)}{\varphi(x+v)} \right) dv + o(1) = o(1), \tag{18}$$

since $1/\varphi(x)$ is bounded. That is, $\tilde{h}(x)$ is slowly varying in the additive Karamata sense (as with $\log \varphi$ in the lemma). So, by the Karamata–de Bruijn representation (see BGT, 1.3.3),

$$\tilde{h}(x) = c(x) + \int_0^x e(v)\,dv, \tag{19}$$

for some measurable $c(\cdot) \to c \in \mathbb{R}$ and continuous $e(\cdot) \to 0$. Rearranging yields

$$\log f(x) = h(x) = \tilde{h}(x) + \rho \int_1^x \frac{du}{\varphi(u)} = c(x) + \rho \int_1^x \frac{du}{\varphi(u)} + \int_0^x e(v)\,dv.$$

Taking $d(x) = e^{c(x)}$, we obtain the desired representation. To check this, without loss of generality we now take $d(x) = 1$, and continue by substituting $u = x + s\varphi(x)$ to obtain, from (18) and (19),

$$\int_x^{x+t\varphi(x)} e(v)\,dv = \tilde{h}(x+t\varphi(x)) - \tilde{h}(x) = o(1),$$

since $\varphi \in \mathrm{SN}$. □

The proof above remains valid when $\rho = 0$ for arbitrary $\varphi \in \mathrm{SN}$, irrespective of whether φ is bounded away from zero or not. Since φ is itself φ-regularly varying with corresponding index $\rho = 0$, we have an alternative to the Bloom–Shea representation of φ via the de Bruijn–Karamata representation. We record this as the following corollary.

Corollary 2. *For measurable $\varphi \in \mathrm{SN}$, there are measurable $d(\cdot) \to d \in (0, \infty)$ and continuous $e(\cdot) \to 0$ such that*

$$\varphi(x) = d(x) \exp\left(\int_1^x e(v)\,dv\right),$$

where, for each t,

$$\int_x^{x+t\varphi(x)} e(v)\,dv = o(1).$$

We have in the course of the proof of Theorem 8 in fact also shown the following corollary.

Corollary 3. *For $\varphi \in \mathrm{SN}$ bounded away from 0 and f measurable and φ-regularly varying:*

$$f(x) = \tilde{f}(x) \exp\left(\rho \int_1^x \frac{du}{\varphi(u)}\right),$$

for some $\rho \in \mathbb{R}$ and some Karamata (multiplicatively) slowly varying $\tilde{f}$.

10. *Complements*

1. *Tauberian Theorems.* Recall (see, for example, [**45**, II.8]) that Wiener's Tauberian theorem is a consequence of Wiener's approximation theorem: that, for $f \in L_1(\mathbb{R})$, the following are equivalent:

(i) linear combinations of translates of f are dense in $L_1(\mathbb{R})$;
(ii) the Fourier transform $\hat{f}$ of f has no real zeros.

The result is the key to Beurling's Tauberian theorem [**45**, IV Theorem 11.1]. Rate-of-convergence results (Tauberian remainder theorems) are also possible; see, for example, [**45**, VII.13].

The theory extends to Banach algebras (indeed, played a major role in their development). In this connection [**45**, V.4], we mention weighted versions of L_1 : for *Beurling weights* ω, positive measurable functions on $\mathbb{R}$ with subadditive logarithms, $\omega(t+u) \leqslant \omega(t)\omega(u)$, define $L_\omega = L_{1,\omega}$ to be the set of f with

$$\|f\| = \|f\|_{1,\omega} := \int_{\mathbb{R}} |f(t)|\omega(t)\,dt < \infty.$$

Then (*Beurling's approximation theorem*): Wiener's approximation theorem extends to the weighted case when ω satisfies the non-quasi-analyticity condition

$$\int_{\mathbb{R}} \frac{|\log \omega(t)|}{1+t^2}\,dt < \infty.$$

This condition has been extensively studied (see, for example, [**44**]) and is important in probability theory (work by Szegő, see, for example, [**7**]).

As mentioned in the Introduction, Beurling's Tauberian Theorem was motivated by the Borel summability method, and its relative, the Valiron method. These are not of convolution form, either additive or multiplicative. As Wiener Tauberian theory deals with convolutions, two options have been used here. First, one can use an approximation procedure, as in Pitt [**61**, §§ 4.3, 4.4]; secondly, one can use Hardy–Littlewood methods long predating Wiener theory, as in [**67**]. Beurling's Tauberian theorem allows an effortless reduction to the Wiener case, and this illustrates the power of Beurling slow variation.

All these Tauberian methods are useful in probability theory, for example, in the context of extensions to the law of large numbers; see, for example, [**8**, **20**].

2. *Infinite combinatorics*. Translations of a null sequence into a non-negligible set go back to Banach [**5**] in the Baire case, and were first studied systematically by Kestelman in connection with additive functions on the reals, and much later by Borwein and Ditor, see § 10.4 and [**54**]. The embedding property of CET is our generalization; this broadens the scope of these ideas to normed groups [**13**] and to group actions (as here with the affine group action on the reals). Other possible settings include semitopological groups, paratopological groups, etc.; see e.g. [**32**]. Underlying it, and pervasive in our work, are topological properties of *shift-compactness* linking group actions (see [**55**] Miller, Miller-Van Wieren and Ostaszewski (in preparation)) and topologically large sets (see, for example, [**16**]). Shift-compactness is an even more powerful tool than the Open Mapping Principle: indeed, the Effros Open Mapping Principle (for which see van Mill [**52**]), concerning openness of transitive group actions on a metric space, is its strongest consequence known to date. Other applications include a short proof simultaneously of the Steinhaus Sum-set theorem (that 0 is in the interior of $A - A$ for non-null measurable A) and of its Baire analogue, the Piccard–Pettis theorem. Indeed, shift-compactness has the ability to reduce hard proofs into easy ones, as here with Bloom's proof of his theorem. By reducing from quantitative measure theory to qualitative measure theory, we open the door to category-measure duality, here.

The application of Theorem CET above requires the verification of (wcc), and in the measure case this calls for just enough of the quantitative aspects to suffice, see [**15**]. The Baire and measure cases come together here via the coincidence between metric and measure for real intervals; cf. § 5.

3. *Beurling theory and Karamata theory*. The Beurling theory developed here (Beurling slow and regular variation) includes Karamata theory (of slow and regular variation, BGT Chapters 1, 2), and thereby also Bojanic-Karamata/de Haan theory (BGT Chapter 3). For, taking $\varphi(x) \equiv 1$ in Part II gives Karamata theory in additive form directly, and hence in multiplicative form also via the log/exp transformation. Also, if we drop the condition $\varphi(x) = o(x)$, and require that φ is *self-equivarying* as in [**59**] (so that $\varphi(x + t\varphi(x))/\varphi(x) \to \lambda(t)$), then $\varphi(x) \equiv x$ is allowed, and one can check that the results of Part II then give Karamata theory in multiplicative form directly. Linking these two cases is the λ-UCT of [**59**] (in the notation there, $\lambda(t) \equiv 1$ corresponds to $\varphi(x) \equiv 1$, $\lambda(t) \equiv 1 + t$ to $\varphi(x) = x$; see (BSV)).

That each of Karamata and de Haan theory contains the other follows from the 'double-sweep' procedure of BGT § 3.13.1.

4. *Miller homotopy*. Given Theorems 8 and 9, without loss of generality in the asymptotic analysis one may take positive $\varphi \in \mathcal{C}^1$, continuously differentiable; then $H(x, t) = x + t\varphi(x)$ satisfies the following three ('homotopy') properties:

(i) $H(x, 0) = x$;
(ii) H_x, H_t exist and
(iii) $H_t(t, x) > 0$;

which enable Miller [**53**] to generalize the Kestelman–Borwein–Ditor above that, *for any non-negligible* (*Baire/measurable*) *set* T *and any null sequence* z_n, *for quasi-all* $t \in T$ *there is an* *infinite* $\mathbb{M}_t \subseteq \mathbb{N}$ *such that* $\{H(t, z_m) : m \in \mathbb{M}_t\} \subseteq T$, that is, in our context, $\{t + \varphi(t)z_m : m \in \mathbb{M}_t\} \subseteq T$; cf. Lemma 2.

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