Interplay between distributional and temporal dependence

An empirical study with high-frequency asset returns

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Summary. The recent popularity of copulas in the analysis and modelling of multivariate financial time series arises from several applications in the financial sector. This paper surveys the most important techniques of modelling and measuring distributional dependence with a view towards financial applications such as pricing and hedging financial instruments and portfolio risk management. The term distributional dependence refers to the (contemporaneous) dependence among multiple time series. The majority of results of the existing statistical literature on copulas assumes i.i.d. data. However, real financial time series incorporate temporal dependence such as volatility clustering or seasonality. Moreover, common filtering techniques, for example (G)ARCH filtering, usually also lead to a rejection of the i.i.d. hypothesis due to model misidentification. In this paper we investigate the sensitivity of (distributional) dependence measures with respect to various filtering techniques utilizing a IBM-GM high-frequency data set. The main focus will be on the distributional dependence of extreme events which is important for risk management. Our results show that filtering techniques crucially affect the distributional dependence structure.

Keywords: Distributional dependence; temporal dependence; copula; Kendall's tau; tail dependence; high-frequency asset returns; GARCH process; autocorrelation function (ACF); multivariate analysis.

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1 Introduction

Copula functions link multivariate distributions to their corresponding univariate marginals and allow one to study the distributional dependence of multivariate distributions. In contrast to temporal dependence of a time series, the term distributional dependence refers to the (contemporaneous) dependence among multiple time series. In finance and insurance, copulas have recently become very popular due to two important applications.

First, copulas have been recognized as a promising tool to analyze and model the dependence structure of credit-risky portfolios [BDNRR00], [FM01], [ELM01], [BKS03]. The adequate modelling of dependence in credit portfolios has been identified as one of the most important and pressing issues to be addressed in modern credit-risk management. This is partly because the pressure of globalization has led to a significant increase of dependencies within assets and asset classes of particular markets and between markets. For example, many empirical studies, such as [KS96], [LS01], and [CKK02], have focused on the so-called "correlation break-down". The latter refers to the significant increases of distributional dependence between financial asset returns during bear markets, *which leads to failure of conventional diversification strategies in times when they are most needed*. In particular, the precise analysis of the extreme (negative) returns of an asset portfolio, which depends heavily on the dependence structure of the individual extreme asset returns, must be studied carefully as it provides important insights into the appropriate supply of economic capital, cf. [O99].

Second, in order to manage and control portfolio credit risk, a new generation of financial instruments such as basket credit derivatives and collateralised debt obligations (CDOs) has been introduced to financial markets. The pricing and hedging of these instruments require a careful analysis of the dependence structure between the respective underlying as well. For the active management of portfolio credit risk, copulas have recently been applied to model the dependence structure between default times involved in the pricing and hedging of basket credit derivatives and CDOs. For example, [L00] utilizes the so-called Gaussian copula to price first-to-default credit derivatives. [LG03] and [S03] extend the copula-based pricing to other basket credit derivatives and CDOs by applying other types of copulas.

For further application, see [DNO92] or [P01] for a time series approach with copulas and [GGW04], who apply copulas in the framework of multidimensional option pricing.

This paper provides a survey of the most important techniques of modelling and measuring distributional dependence with a view towards pricing and hedging the afore-mentioned financial instruments and towards portfolio risk management. In the first section we present the concept of copulas and relevant results, and we outline their importance for analyzing distributional dependence. In passing we introduce the family of tail copulas which helps analyzing the distributional dependence of extreme events. We then discuss various dependence measures related to (tail) copulas and discuss their financial applications. Afterwards we focus on nonparametric statistical inference for (tail) copulas and dependence measures and point out that the majority of statistical results are valid under the assumption of i.i.d. data. However, it is well known that every real financial time series incorporates temporal dependence. For example, we will show that high-frequency financial data may possess a very characteristic seasonal and autoregressive temporal dependence structure of its volatilities. The latter is often referred to as volatility clustering. The amount of literature on filtering techniques for time series, in order to obtain i.i.d. data, is enormous. In practice, however, the most popular filtering technique for volatility clustering of asset returns is unquestionably the (G)ARCH filtering. Although (G)ARCH filtering usually leads to a rejection of the i.i.d. hypothesis of the resulting residuals due to model misidentification, its simple interpretation, estimation and forecasting has made it the favorite filtering technique in the financial industry. (G)ARCH models have been introduced and discussed in [B86], [DG96], and [A01].

The second part of the paper continues with the previous discussion and investigates the sensitivity of distributional dependence measures towards deseasonalisation and GARCH filtering for a General Motors (GM) and International Business Machines (IBM) high-frequency data set. Our particular choice of the GARCH filter is justified by its afore-mentioned popularity. High-frequency data are of interest because of several so-called stylized dependence facts. We will especially focus on the distributional dependence of extreme events.

Our results show that filtering techniques crucially affect the distributional dependence structure and thus inherit the danger of wrong conclusions from inappropriate dependence measures. As a side product we advocate autocorrelation functions (ACF) based on scale-invariant (copula-based) dependence measures and provide new insights into the interplay between distributional and temporal dependence of multivariate time series. The discussion of a new type of nonparametric estimator for the so-called tail dependence gives insight into the dependence measurement of extreme events. We will compare our results with the findings of [BDE03].

2 Modelling distributional dependence

Each multivariate distribution function can be split into its univariate marginal distribution functions and a copula function (Sklar's theorem, [S59]). In other words, copulas allow one to study the distributional dependence structure of random vectors irrespective of their marginal distributions.

Definition 1 (Copula). *Let $X = (X_1, \ldots, X_d)'$ be an d-dimensional random vector with distribution function $F(x_1, \ldots, x_d) = P(X_1 \le x_1, \ldots, X_d \le x_d)$ and marginal distribution-functions $F_i(x_i) = P(X_i \le x_i)$ for all $i =$*

$1,\ldots,d$. *Then the distribution function* C *of the* d*-dimensional random vector* $(F_1(X_1),\ldots,F_d(X_d))'$ *is called copula (or copula function) of* X *or* F.

It can be shown that the copula function is uniquely determined by the multivariate distribution function F if all univariate marginal distribution functions are continuous (Sklar's Theorem) and that

$$F(x_1,\ldots,x_d) = C(F_1(x_1),\ldots,F_d(x_d)).$$

Thus, copulas can be utilized to build flexible multivariate distribution functions in two steps: First, model the distributional dependence via some copula, and second, plug in appropriate marginals.

Copula functions represent standardized distributions in the sense that their one-dimensional marginals are uniformly distributed on the interval $[0,1]$. An important property is that the copula of a random vector X stays the same irrespectively of any strictly increasing transformation of the marginals $X_j,\ j=1,\ldots,d$. This invariance property (also called "scale invariance") is a desired feature of dependence functions and dependence measures, as we understand dependence itself to represent the association between "large" and "small" realizations of random vectors irrespectively of their scale.

Kendall's tau and Spearman's rho. A proper dependence measures for multivariate distributions should be scale invariant (or invariant under change of the marginal distributions). All dependence measures derived from the copula are scale invariant, and so in line with our basic requirement. The most important scale invariant dependence measure in financial applications is Kendall's τ.

Definition 2 (Kendall's tau). *Let* X *and* $\bar{X}$ *be independent* d*-dimensional random vectors with common continuous distribution function* F *and copula* C. *Kendall's tau of the margins* X_i *and* $X_j,\ i<j$, *is defined by*

$$\begin{aligned}\tau_{ij} :&= \mathbb{P}((X_i-\bar{X}_i)(X_j-\bar{X}_j)>0) - \mathbb{P}((X_i-\bar{X}_i)(X_j-\bar{X}_j)<0)\\ &= 4\int_{[0,1]^2} C_{ij}(u_i,u_j)\,dC_{ij}(u_i,u_j) - 1, \qquad (1)\end{aligned}$$

where $C_{ij}(u_i,u_j) = C(1,\ldots,1,u_i,1,\ldots,1,u_j,1\ldots,1)$.

The finite-sample version of Kendall's tau $\hat{\tau}_{ij}$ is defined as the ratio of the number of concordant minus the number of discordant pairs of sample points with respect to the number of concordant and discordant pairs of sample points. Here, a pair of sample points (x_i,x_j) and $(\bar{x}_i,\bar{x}_j)$ is called concordant if $x_i<(>)\bar{x}_i$ and $x_j<(>)\bar{x}_j$, and discordant otherwise. Formally

$$\hat{\tau} = \frac{\text{concordant pairs } - \text{ disconcordant pairs}}{\text{concordant pairs } + \text{ disconcordant pairs}}. \qquad (2)$$

Obviously this dependence measure is scale-invariant and it represents one of the most intuitive dependence measures.

The Pearson's correlation coefficient $\rho(X_i, X_j)$ of the i-th and j-th component of $X = (X_1, \dots, X_d)'$ measures linear dependence and is thus not scale-invariant. However, we might intuitively substitute for the random variables X_i and X_j the standardized random variables $F_i(X_i)$ and $F_j(X_j)$ in order to obtain the scale-invariant correlation coefficient $\rho(F_i(X_i), F_j(X_j))$. Indeed, this dependence measure is well known and is called Spearman's rho $\rho^S_{ij} := \rho(F_i(X_i), F_j(X_j))$. It can be shown that

$$\rho^S_{ij} = 12 \iint_{[0,1]^2} C_{ij}(u_i, u_j)\, du_i du_j - 3.$$

In contrast to Pearson's correlation coefficient, the latter two dependence measures are always 1 or -1, respectively, if one random variable is an increasing function (completely positively correlated) or decreasing function (completely negatively correlated) of the other. Recall that Pearson's correlation coefficient might be zero in both cases. A detailed treatment of copulas and other dependence measures can be found in [J97] and [N99].

Tail dependence and tail copula. In contrast to the dependence measures discussed so far, tail dependence focuses solely on the distributional dependence of extreme or tail events. In the context of tail dependence, the immediate analogue to copulas, which describe the entire distributional dependence structure, is given by tail copulas. In this paper we restrict ourself to so-called lower tail copulas. However, the results hold similarly for upper tail copulas; see [SS03] for the definition. If not otherwise stated, we assume continuous marginal distributions.

Definition 3 (Tail copula). *Let F be a d-dimensional distribution function with copula C. If for the subsets $I, J \subset \{1, \dots, d\}$, $I \cap J = \emptyset$, the following limit exists everywhere on $\bar{\mathbb{R}}^d_+ := [0,\infty]^d \backslash \{(\infty, \dots, \infty)\}$:*

$$\begin{aligned} \Lambda^{I,J}_L(x) &:= \lim_{t\to\infty} \mathbb{P}(X_i \le F_i^{-1}(x_i/t),\ \forall i \in I \mid X_j \le F_j^{-1}(x_j/t),\ \forall j \in J) \\ &= C(x_i/t,\ \forall i \in I \mid x_j/t,\ \forall j \in J), \end{aligned} \tag{3}$$

then the function $\Lambda^{I,J}_L : \bar{\mathbb{R}}^d_+ \to \mathbb{R}$ is called a lower tail-copula associated with F (or C) with respect to I, J.

For simplicity and notational convenience all further definitions and results are provided only for the bivariate case. The multidimensional extensions are given in [SS03]. The statistical inference becomes easier if the following slight modification of the tail copula is utilized:

$$\Lambda_L(x_1, x_2) := x_2 \cdot \Lambda^{\{1\},\{2\}}_L(x_1, x_2), \quad x_1 \in \bar{\mathbb{R}}_+, x_2 \in \mathbb{R}_+, \tag{4}$$

where the indices $\{1\}$ and $\{2\}$ can be dropped. Further, set $\Lambda_L(x_1, \infty) := x_1$ for all $x_1 \in \mathbb{R}_+$.

The next definition embeds the well-known tail-dependence coefficient (see [J97], p. 33) within the framework of tail copulas.

Definition 4 (Tail dependence). *A bivariate random vector $(X_1, X_2)'$ is said to be lower tail-dependent if $\Lambda_L(1,1)$ exists and*

$$\lambda_L := \Lambda_L(1,1) = \lim_{v \to 0^+} \mathbb{P}(X_1 \le F_1^{-1}(v) \mid X_2 \le F_2^{-1}(v)) > 0. \tag{5}$$

Consequently, $(X_1, X_2)'$ is called lower tail-independent if λ_L equals 0. *Further, λ_L is referred to as the lower tail-dependence coefficient.*

It is well known that the multivariate normal distributions, the multivariate generalized-hyperbolic distributions, and the multivariate logistic distributions are lower tail-independent whereas the multivariate t-distributions and the α-stable distributions are lower tail-dependent. For a general account on tail dependence for elliptically-contoured distributions we refer to [S02]. Both preceding definitions show that tail dependence is again a copula property. In particular, the tail-dependence coefficients are invariant under strictly increasing transformations of the marginals.

Practitioners interpret tail dependence as the limiting likelihood of an asset/portfolio return falling below its Value at Risk given that another asset/portfolio return has fallen below its Value at Risk.

Application: CDOs and multi-name credit derivatives. We have already mentioned in the introduction of this paper that the increasing active management and control (in contrast to the traditional passive management and control) of portfolio credit risk has led to a new generation of financial instruments such as multi-name credit derivatives and collateralised debt obligations (CDOs). Examples of these instruments are basket credit default swaps (We refer to [BOW03] for more background reading.). Because of the association with a pool of credit-risky underlying, the pricing and hedging of these instruments require a careful analysis of the dependence structure between the respective underlying. In this context, copulas have recently been applied to model the dependence structure between *default times* of the underlying. Let us consider a portfolio of d underlying assets and let τ_i represent the default time of the ith underlying (or the corresponding obligor). Further, let $F_i(t) = \mathrm{P}(\tau_i \le t)$ be the marginal distributional function of the default time of obligor i. The copula function C is now used to obtain the multivariate default-time distribution $F(t_1, \ldots, t_d) = C(F_1(t_1), \ldots, F_d(t_d))$. The latter approach allows to calibrate the default-time distribution, in the first step, for each margin separately. This calibration is equivalent to the construction of a so-called credit yield curve (We would like to point out that the banking sector uses already quite sophisticated construction methods.). In the second step, a parametric copula is usually calibrated via some scale-invariant dependence measure such as Kendall's tau. The optimal choice of the copula is the topic of many recently published research papers. For example, [L00] utilizes the so-called Gaussian copula to price first-to-default credit derivatives. [LG03] and [S03] extend the copula-based pricing to other basket credit derivatives and CDOs by applying other types of copulas.

3 Statistical inference

Empirical copula. Concerning the estimation of copula functions, several parametric, semi-parametric, and nonparametric procedures have already been proposed in the literature (cf. [S84], [GR93], [GGR95]). Regarding the nonparametric estimation, [D79], [D81], and [FRW02] establish weak convergence of the so-called *empirical copula process* under independent and dependent marginal distributions. In the following we will confine to the bivariate case.

Definition 5 (Empirical copula). *Consider the bivariate random sample* $\{(X_i^1, X_i^2)', \ i = 1, \ldots, n\}$. *Then the corresponding empirical copula is defined by*

$$C_n(u_1, u_2) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{F_{1,n}(X_i^1) \leq u_1, F_{2,n}(X_i^2) \leq u_2\}, \tag{6}$$

where $\mathbf{1}$ *denotes the indicator function and* $F_{j,n}, \ j = 1, 2$ *is* $n/(n+1)$ *times the empirical distribution function of* $X^j, \ j = 1, 2$.

Note that the empirical copula is a function of the ranks of the observations. Powerful test for independence or goodness of fit (such as Cramér-von Mises or Kolmogorov-Smirnov) could be based on functionals of the empirical copula. However, there does not exists a simple expression for the asymptotic distribution of the *empirical copula process*

$$\mathbb{C}_n(u_1, u_2) = \sqrt{n}\{C_n(u_1, u_2) - C(u_1, u_2)\}. \tag{7}$$

The limiting process of (7) is derived in [S84] and [GS87] (Test of independence based on the empirical copula process can be found in [GR03].). Analogous limiting results, although one needs different techniques of proof, can be obtained for the so-called empirical tail copula process.

Empirical tail copula. A nonparametric estimator, the so-called empirical tail copula, for the bivariate lower tail-copula $\Lambda_L(x_1, x_2)$, $(x_1, x_2)' \in \bar{R}_+^2$, is proposed. Note that nonparametric estimation turns out to be appropriate for unknown tail copulas as no general finite-dimensional parametrization of tail copulas exists (in contrast to the one-dimensional extreme value distributions). The choice of the empirical distribution function to model the marginal distribution avoids any misidentification of the copula due to a wrong parametrical fit of the marginal distributions. Empirical investigations regarding such misidentifications and misinterpretations of the corresponding (extremal) dependence structure are provided in [FJS03].

Definition 6 (Empirical tail copula). *Consider the bivariate random sample* $\{(X_i^1, X_i^2)', \ i = 1, \ldots, n\}$ *and denote the rank of* X_i^1 *and* X_i^2 *by* $\mathcal{R}_{in}^1$ *and* $\mathcal{R}_{in}^2$, *respectively. The lower empirical tail copula is defined via formula (3) by:*

$$\hat{\Lambda}_{L,n}(x_1,x_2) := \frac{n}{k} C_n\Big(\frac{kx_1}{n}, \frac{kx_2}{n}\Big) = \frac{1}{k}\sum_{i=1}^{n} \mathbf{1}\{\mathcal{R}^1_{in} \le kx_1 \text{ and } \mathcal{R}^2_{in} \le kx_2\}$$

with empirical copula C_n and some threshold $k \in \{1,\ldots,n\}$.

The optimal choice of the threshold k in Definition 6 is related to the usual variance-bias problem known from tail index estimations of regular varying distribution functions, and will be addressed in a forthcoming work. For the asymptotic results we assume that $k = k(n) \to \infty$ and $k/n \to 0$ as $n \to \infty$.

Definitions 5 and 6 can be generalized for bivariate time series. In this case we refer to the empirical (tail) copula as *quasi-empirical (tail) copula.*

Condition 1 (Second-Order Condition) *The lower tail-copula $\Lambda_L(x,y)$ is said to satisfy a second-order condition if a function $A : \mathbb{R}_+ \to \mathbb{R}_+$ exists such that $A(t) \to 0$ as $t \to \infty$ and*

$$\lim_{t\to\infty} \frac{\Lambda_L(x,y) - tC(x/t, y/t)}{A(t)} = g(x,y) < \infty$$

locally uniformly for $(x,y)' \in \bar{\mathbb{R}}^2_+$ and some nonconstant function g. The second-order condition for the upper tail-copula is defined analogously.

Note that $A(t)$ is regularly varying at infinity so this is just a second-order condition on regular variation, cf. [DS96].

Theorem 2 (Asymptotic normality). *Let F be the bivariate distribution function of the random sample $\{(X^1_i, X^2_i)',\ i = 1,\ldots,n\}$ with continuous marginal distribution functions F_1 and F_2. If the tail copula $\Lambda_L \not\equiv 0$ exists, possesses continuous partial derivatives, and the Second-Order Condition 1 holds, then for $n \to \infty$*

$$\sqrt{k}\big\{\hat{\Lambda}_{L,n}(x_1,x_2) - \Lambda_L(x_1,x_2)\big\} \xrightarrow{w} \mathbb{G}_{\Lambda_L}(x_1,x_2),$$

where $\mathbb{G}_{\Lambda_L}(x_1,x_2)$ is a centered tight continuous Gaussian random fields. Weak convergence takes place in the space of uniformly-bounded functions on compacta in $\bar{\mathbb{R}}^2_+$. The covariance structure of $\mathbb{G}_{\hat{\Lambda}_L}(x_1,x_2)$ is given in Corollary 1 below.

Corollary 1 (Covariance structure). *The limiting process in Theorem 2 can be expressed by*

$$\begin{aligned}\mathbb{G}_{\hat{\Lambda}_L}(x_1,x_2) &= \mathbb{G}_{\hat{\Lambda}^*_L}(x_1,x_2) \\ &\quad - \frac{\partial}{\partial x_1}\Lambda_L(x_1,x_2)\mathbb{G}_{\hat{\Lambda}^*_L}(x_1,\infty) - \frac{\partial}{\partial x_2}\Lambda_L(x_1,x_2)\mathbb{G}_{\hat{\Lambda}^*_L}(\infty,x_2),\end{aligned} \tag{8}$$

where $\mathbb{G}_{\Lambda_L}(x_1,x_2)$ is a centered tight continuous Gaussian random field. The covariance structure of $G_{\Lambda^*_L}$ is given by

$$\mathbb{E}\big(\mathbb{G}_{\hat{\Lambda}^*_L}(x_1,x_2)\cdot \mathbb{G}_{\hat{\Lambda}^*_L}(\bar{x}_1,\bar{x}_2)\big) = \Lambda_L\big(\min\{x_1,\bar{x}_1\}, \min\{x_2,\bar{x}_2\}\big) \tag{9}$$

for $(x_1,x_2)', (\bar{x}_1,\bar{x}_2)' \in \bar{\mathbb{R}}^2_+$.

The proof of asymptotic normality is accomplished in two steps. In the first step the marginal distribution functions F_1 and F_2 are assumed to be known and an asymptotic normality result is derived. In the second step the marginal distribution functions F_1 and F_2 are assumed to be unknown and the asymptotic result is proven by utilizing a particular version of the functional delta method, as provided in [VW96].

The evaluation of the empirical tail copula at the point $(1,1)'$ immediately yields a non-parametric estimator for the lower tail-dependence coefficient. The estimation of the lower tail-dependence coefficient (briefly: lower TDC) is important for practical applications, for example in risk management where one is primarily interested in the dependence between large loss events, and has been addresses in several publications, see [MS02], [JM02], [AK03], [BDE03], and [FJS03]. Consider the following nonparametric estimator for the lower TDC:

$$\hat{\lambda}_{L,n}(k) = \hat{\Lambda}_{L,n}(1,1) = \frac{1}{k} \cdot \sum_{j=1}^{n} \mathbf{1}\{\mathcal{R}_{in}^{1} \leq k \ \wedge \ \mathcal{R}_{in}^{2} \leq k\} \qquad 1 \leq k \leq n,$$

with $k = k(n) \to \infty$ and $k/n \to 0$ as $n \to \infty$.

Under the same technical conditions as in Theorem 2 we obtain that

$$\sqrt{k}\big\{\hat{\lambda}_{L,n} - \lambda_L\big\} \xrightarrow{d} \mathbb{G}_{\lambda_L},$$

with $\mathbb{G}_{\lambda_L}$ being centered and normally distributed, i.e. $\mathbb{G}_{\lambda_L} \sim N(0, \sigma_L^2)$ with

$$\begin{aligned}\sigma_L^2 = \lambda_L &+ \Big(\frac{\partial}{\partial x}\Lambda_L(1,1)\Big)^2 + \Big(\frac{\partial}{\partial y}\Lambda_L(1,1)\Big)^2 \\ &+ 2\lambda_L\Big(\Big(\frac{\partial}{\partial x}\Lambda_L(1,1) - 1\Big)\Big(\frac{\partial}{\partial y}\Lambda_L(1,1) - 1\Big) - 1\Big).\end{aligned}$$

[SS03] prove strong consistency of $\hat{\lambda}_{L,n}$ and $\hat{\Lambda}_{L,n}$ if $k/\log\log n \to \infty$ as $n \to \infty$.

4 Dependence of high-frequency asset returns - An empirical study

4.1 The GM-IBM high-frequency data set

So far we have surveyed important techniques of modelling and measuring distributional dependence for financial time series. We have mentioned the concept of empirical (tail) copulas which is a central element for nonparametric statistical inference from real data. We pointed out that the related results on asymptotic normality and strong consistency are proven under the assumption of i.i.d. data (Note that the limiting distributions are already quite complicated in this case.). However, each financial time series incorporates temporal dependence, i.e. the data cannot be assumed to be independent and identical distributed. Furthermore, almost all common filtering techniques will lead to a rejection of the i.i.d. hypothesis due to the usual model misidentification.

The question is therefore: How sensitive is the distributional dependence (although the measurements are always obtained from data which are temporally dependent) towards various filtering methods?

To give a partial answer to this question we consider a typical financial time series, namely a General Motors (GM) and International Business Machines (IBM) high-frequency data set. High-frequency asset return data comprise several very characteristic dependence features which are usually only found in experimentally-generated time series, and thus they are very interesting for our empirical analysis. Many authors have already been attracted to explore these features. In the framework of univariate time series, [BS02], [ABDL01], [MM01], and [AT02] investigate the estimation of the actual volatility of stochastic-volatility models (SV) by means of so-called realized volatilities of high-frequency data. Further, [AB97], [AB98], [DGMOP01], and [MCT02] address the question of how to model the characteristic (volatility) seasonality and volatility clustering effects of high-frequency data. The direct fitting of well-established financial models to high-frequency asset returns is usually complicated, due to market microstructure effects such as discreteness of prices, bid/ask bounce, irregular trading etc. (see for example [BRT00]). Moving-average structures for asset returns, which often occur as the result of no-trading effects or bid/ask bounce effects, are discussed in [CLM97].

However, there is not much literature on multivariate aspects related to high-frequency financial data; among them we mention [BS03] and [BDE03].

The plan of our statistical analysis. In the first step, we apply various filtering techniques to the afore-mentioned data set in order to obtain approximately i.i.d. data. In particular, we utilize a GARCH filter, in order to reduce the observed volatility clustering of the asset returns, as it is the most popular and common filtering technique in the financial sector. In the second step, we analyze the effect of the filtering on the quasi-empirical copula and on the magnitude of tail dependence. In passing, we introduce autocorrelation functions (ACFs) based on Kendall's tau.

The data. The data of high-frequency asset returns we utilize in this paper correspond to the cleaned bivariate stock prices of GM and IBM over the time horizon 4th of January 1993 to 29th of May 1998. For reasons of market efficiency, we consider 15-minute price quotes which are aggregated from tick-by-tick price quotes leading to a sample size of $n = 36855$ data. The prices are observed each trading day during the time from $9.30h$ to $16.00h$. Figure 1 illustrates the log-return movements over different time intervals.

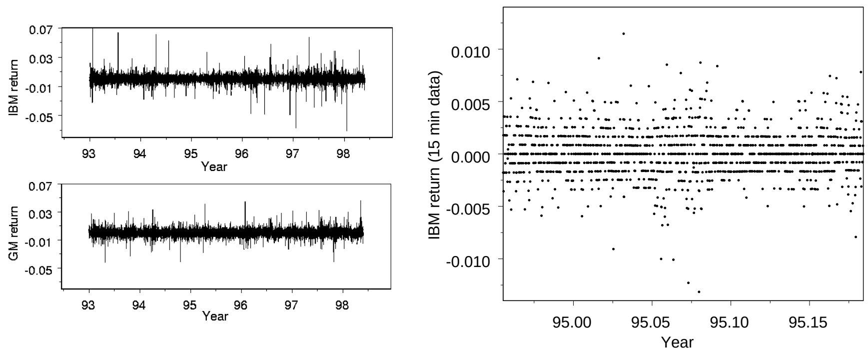

Fig. 1. Stock log-returns for each 15 minutes for General Motors (GM) and International Business Machines (IBM) over the years 1993-1998 (left plot) and over January and February 1995 (right plot).

The *price quotes* are denoted by $P_i^j,\ i = 1, \ldots, n,\ j \in \{GM, IBM\}$ and the corresponding *log-returns* (briefly: returns) are defined by

$$R_i^j := \log(P_i^j) - \log(P_{i-1}^j), \quad i = 2, \ldots, n, \text{ and } R_1^j = 0,\ \ j \in \{GM, IBM\}.$$

The right plot of Figure 1 zooms into the IBM return series at the beginning of the year 1995 and reveals that the volatility clustering is less pronounced than it is typically seen in foreign-exchange (FX) high-frequency data, cf. [BDE03]. The volatility clusters are hardly observable solely by glancing at the plot, and so we provide the *autocorrelation function* (ACF) for the returns R_i^j, the squared returns $(R_i^j)^2$, and the absolute returns $|R_i^j|$, respectively, in Figure 2. Although, the characteristic trading pattern of almost discrete changes of the price quote can be clearly seen in the right plot of Figure 1.

From Figure 2 we learn that the returns themselves are not autocorrelated, but the squared and especially the absolute returns show significant serial and seasonal autocorrelation which is persistent over time. In particular, the time series is not stationary. The latter seasonality has its origin in the contrast between the beginning of the trading day, which shows high volatility, and the middle, which shows low volatility. Figure 3 illustrates the average volatility

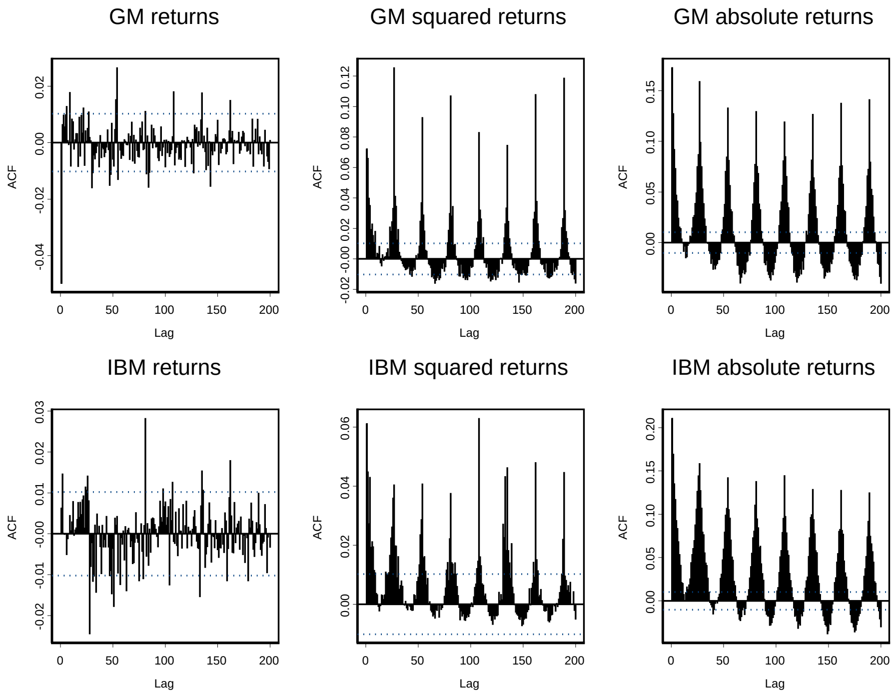

Fig. 2. Autocorrelation function (ACF) for the returns R_i^j (left plots), squared returns $(R_i^j)^2$ (middle plots), and absolute returns $|R_i^j|$ (right plots) for GM and IBM over the years 1993-1998 with lags ranging between 1 and 200.

over the trading day for the return series of GM and IBM. Note that from an economical point of view, the asset returns at $9.30h$ accumulate much more information than the consecutive 15-minute returns. Thus, the $9.30h$ returns are often excluded from the data investigation. However, since our primary interest lies in the dependence structure and not in the economic interpretation, we keep the $9.30h$ data in our analysis.

The immediate problem arising from the latter empirical observations is how to *deseasonalize* the data with respect to the observed volatility structure. Two different approaches are frequently used. We may either utilize the concept of *random time-change*, as described in [DGMOP01] (which preserves additivity of the returns over different time intervals), or we may use *volatility weighting* as in [AB97], [AB98], [MCT02], or [BDE03]. In the latter framework, the deseasonalized returns $\tilde{R}_i^j$ are expressed by

$$\tilde{R}_i^j := c + R_i^j / v_i^j, \quad i = 1, \ldots, n, \ j \in \{GM, IBM\},$$

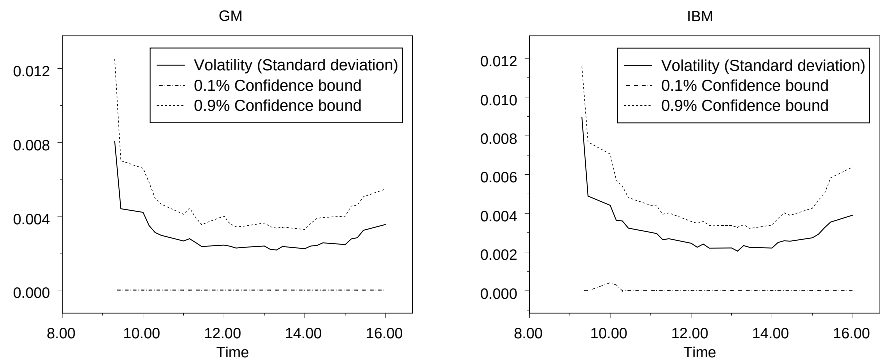

Fig. 3. Volatilities measured by the sample standard-deviation and corresponding empirical confidence bounds over the trading day for returns of GM and IBM over the years 1993-1998.

where $v_i^j,\ i = 1, \ldots, n,\ j \in \{GM, IBM\}$ denote the (expected) seasonal volatilities and c refers to the mean return. The latter volatilities could be derived via some filtering technique from time series theory. A simple approach which is often applied (see for example [BDE03]) estimates the squared volatilities $(v_i^j)^2$ by

$$(v_i^j)^2 = \frac{1}{n_\tau} \sum_{k=1}^{n_\tau} \left(R^j_{k \cdot \tau(i)}\right)^2 \quad j \in \{GM, IBM\},$$

where $\tau(i) = i \text{mod}(1\text{day}) \in \{1, \ldots, 27\}$, since we consider 27 observation times (from $9.30h$ to $16.00h$ in 15-minute steps) per day, and $n_\tau = [n/27]$. The ACF plots for the deseasonalized returns $\tilde{R}_i^j$, provided in Figure 4, illustrate that this approach removes the seasonality of the volatility quite well. However, the lagged volatilities are still serially correlated, and show the typical volatility clustering effect. Note that the absolute returns indicate the characteristic pattern of long-range dependence.

Remark. As with the above marginal volatility weighting, we may weight the bivariate return-vector by the expected seasonal volatility matrix. Although the latter technique seems to be more appropriate for multidimensional data modelling, the main results of this empirical study stay the same.

Finally, we reduce the remaining serial correlation of the volatilities of the deseasonalized returns $\tilde{R}_i^j$ by fitting an univariate GARCH(1,1) model (see [B86]) to each margin separately. Indeed, the GARCH(1,1) models is the most frequently applied GARCH model in practice. Alternatively we fit a multivariate GARCH model to the bivariate deseasonalized return series.

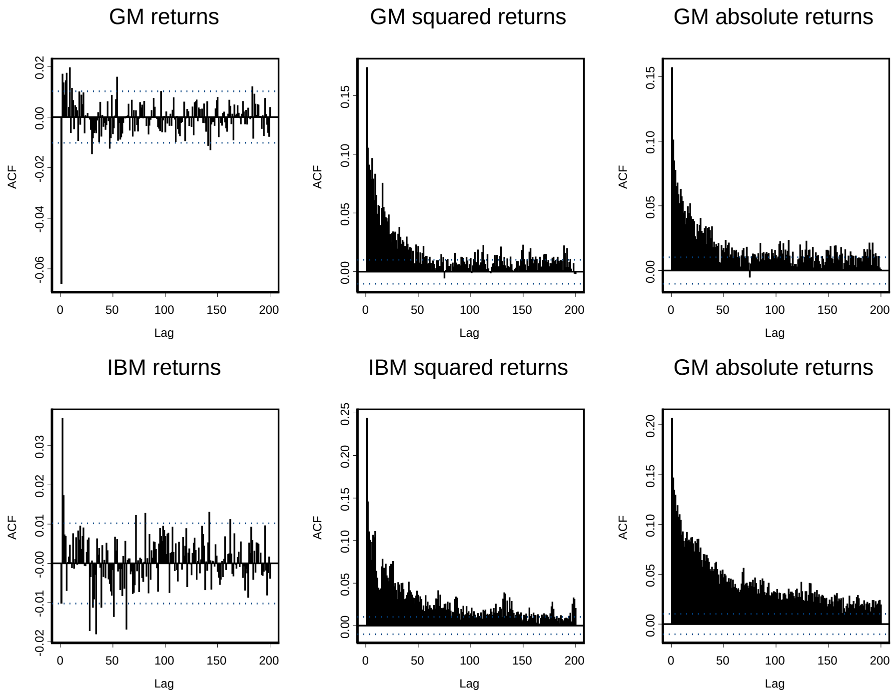

Fig. 4. ACF for the volatility-weighted returns $\tilde{R}_i^j$ (left plots), squared returns $(\tilde{R}_i^j)^2$ (middle plots) and absolute returns $|\tilde{R}_i^j|$ (right plots) for GM and IBM over the years 1993-1998 with lags ranging between 1 and 200.

Regarding the latter, we utilized a diagonal VEC(1,1) model (DVEC(1); see [BEW88]) for the deseasonalized returns $\tilde{R}_i^j$. Both models assume the following covariance dynamics:

$$\Sigma_i = A + B \otimes (\epsilon_{i-1}\epsilon_{i-1}') + C \otimes \Sigma_{i-1},$$

where the symbol $\otimes$ stands for the Hadamard product (element-by-element multiplication) and $A, B, C \in I\!R^{2\times 2}$ (in the univariate case, B and C are diagonal matrices). To improve our fit, we model the error terms ϵ_i via a bivariate Student t-distribution.

Although after each GARCH filtering we must reject the hypothesis of i.i.d. residuals, the ACFs of the residual's covariances imply that the serial correlation of the cross-correlations is not that significant any more. It turns out that the residuals themselves are slightly autocorrelated over the first lag of 15min; however, this time frame is too short for significant arbitrage opportunities.

Remark. According to our empirical study, the main results stay the same irrespective of the choice of a multivariate or an univariate GARCH model.

4.2 Excursion: Analyzing the temporal dependence with Kendall's tau

In Figures 2 and 4 we analyzed the ACF to draw conclusions about the temporal dependence of the underlying (volatility weighted) asset returns. Especially Figure 2 indicates that there might be an unusually large dependence between the return data with a lag of k-days (i.e. lag= $k \cdot 27$). Undoubtedly there is a larger dependence at this special lag, but the correlation coefficient, which can only measure linear dependence, exaggerates the magnitude enormously. A standardization of the bivariate return data to approximately uniformly distributed margins (via the quasi-empirical distribution function which is again explained in formula (10) below) gives a better picture of the respective serial dependence. Figure 5 shows that all large peaks in the ACF disappear after this standardization. The sensitivity of the correlation coefficient under monotone increasing transformations is thus misleading as to the proper analysis of the temporal dependence structure. This is especially so if the dependence is non-linear, as it is in our case. As an alternative, we advocate a new *ACF based on the scale-invariant dependence measure Kendall's tau.* Note that the definition of Kendall's tau requires a common continuous distribution function; however, the corresponding marginal distribution functions might be discontinuous.

Definition 7 (ACF based on Kendall's tau). *Let $(Y_i)_{i\in\mathbb{N}}$ denote a sequence of random variables (or univariate time series). The autocorrelation with lag j of some $Y_i,\ i=j+1,\ldots$ based on Kendall's tau is defined by*

$$\tau_j = \mathbb{P}((Y_i-\bar{Y}_i)(Y_{i-j}-\bar{Y}_{i-j})>0) - \mathbb{P}((Y_i-\bar{Y}_i)(Y_{i-j}-\bar{Y}_{i-j})<0),$$

where $(\bar{Y}_i,\bar{Y}_{i-j})'$ is an independent copy of $(Y_i,Y_{i-j})'$ which has a common continuous distribution function. The plot of τ_j against j is called the ACF based on Kendall's tau.

The sample autocorrelation with lag j based on Kendall's tau is defined as the sample version of Kendall's tau derived from the realizations of $(Y_i,Y_{i-j})',\ i=j+1,\ldots,n$ (see formula (2)).

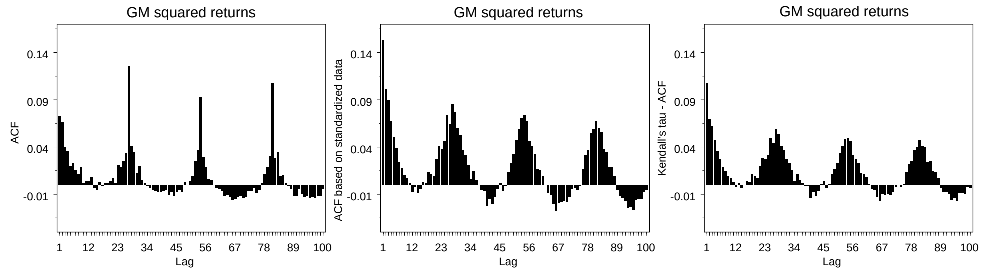

Fig. 5. ACF for the squared returns $(R_i^{GM})^2$ (left plot), squared returns which are standardized by the quasi-empirical distribution function (middle plot) and ACF based on Kendall's tau ACF the squared returns $(R_i^{GM})^2$ over the years 1993-1998 with lags ranging between 1 and 100.

4.3 Analyzing the quasi-empirical copula

We return to our question:

How much did we change the distributional dependence structure?

Let $\{(X_i^1, X_i^2)',\ i = 1, \ldots, n\}$ denote some bivariate time series. Consider the transformed series

$$\{(F_{1,n}(X_i^1), F_{2,n}(X_i^2))',\ i = 1, \ldots, n\}, \tag{10}$$

where $F_{j,n},\ j = 1, 2$, is $n/(n+1)$ times the quasi-empirical distribution function of $X^j,\ j = 1, 2$. We apply transformation (10) to the original GM-IBM returns R_i^j, to the volatility-weighted returns $\tilde{R}_i^j$, and to the GARCH residuals of the volatility-weighted returns.

The results are illustrated in Figure 6. Note that only for the third data set, the underlying data are approximate realization of an empirical copula since these data are closest to i.i.d. For the second data set, the volatility weighted returns, we could impose some ergodicity or mixing conditions to ensure the weak convergence of the quasi-empirical copula to the corresponding real copula (see for example [DMS02]). The latter seems to be not possible for the first data set because the time series is not even stationary. However, transformation (10) gives a better indication of the underlying distributional dependence structure than, for example, a simple scatter plot. Although, any interpretations from related dependence measures should be considered very carefully.

The left plots of Figure 6 illustrate the returns R_i^j, the volatility weighted returns $\tilde{R}_i^j$, and the GARCH residuals of the volatility weighted returns of GM and IBM after they have been transformed (or standardized) according to formula (10). These plots refer to the quasi-empirical copula density. The characteristic cross in the middle of the two upper-left plots indicates the atomic mass of zero returns; i.e. time points where the stocks are not traded.

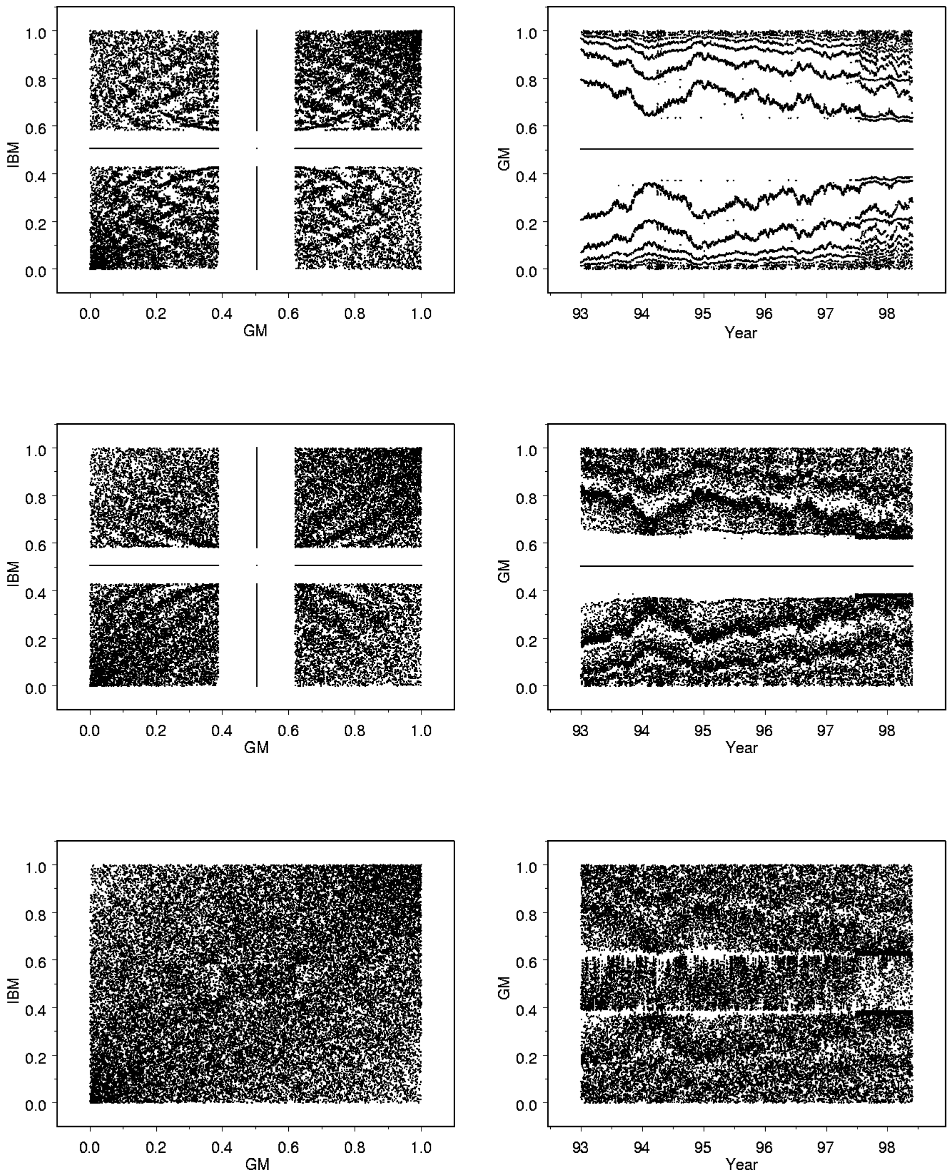

Fig. 6. Quasi-empirical copula density (left plots) of the returns R_i^j (upper plots), the volatility-weighted returns $\tilde{R}_i^j$ (middle plots), and the GARCH residuals of the volatility-weighted returns $\tilde{R}_i^j$ (lower plots) for GM and IBM over the years 1993-1998 and corresponding transformed margins $\hat{F}_1(R_i^{GM})$ (right plots).

Note that the copula is not uniquely defined for discontinuous distribution functions. All other modes of the marginal return distributions, which have been present in Figure 1, are not observable in this plot, which shows that the latter transformation really removes the characteristics of the marginal distributions. We would like to point out the intensifying accumulation of data points in the lower-left and upper-right corner of all quasi-empirical copula density plots. This feature might be an indicator for tail dependence or, in other words, dependence of extreme events. In the next section we solely concentrate on the problem of whether tail dependence changes heavily after filtering. Note, that the quasi-empirical copula density of the GARCH residuals does not possess the characteristic cross.

The plots on the right side of Figure 6 indicate the evolvements of the transformed margin $\hat{F}_1(R_i^{GM})$ which correspond to the respective quasi-empirical copula density on the left side. The strong impact of the filtering becomes quite clear in these plots. For example the characteristic trading pattern of discrete percentual changes of the price quotes, as illustrated by the lines in the upper-right plot (see also Figure 1), vanish completely after the filtering.

Summarizing the observations, Figure 6 clearly shows that the distributional dependence structures, measured via the quasi-empirical copula, differ completely from each other. This indicates that the filtering has a **strong impact** on the analysis of distributional dependence and on the interpretational power of common dependence measures. *Wrong or misleading economic interpretations* can be drawn, if no attention is paid to this basic insight (see also [FS04] for further statistical pitfalls in dependence modelling). In order to underpin the so-far obtained conclusions, we discuss the impact of filtering on the estimation of tail dependence.

4.4 Analyzing the tail dependence

Because of the complicated temporal-dependence structure of the considered GM-IBM high-frequency asset returns, we favor an estimator which does not depend on any distributional assumptions.

Figure 7 illustrates the estimates $\hat{\lambda}_{L,n}(k)$ of the lower tail-dependence coefficient (TDC) λ_L for various thresholds k for the returns R_i^j, the volatility-weighted returns $\tilde{R}_i^j$, and the GARCH residuals of the volatility-weighted returns of GM and IBM over the years 1993-1998. According to the regular variation property of tail-dependent distributions (see [SS03] for more details), tail dependence is present in a bivariate i.i.d. data set if the plot of $\hat{\lambda}_{L,n}(k)$ for various thresholds k shows a characteristic plateau for small k. This characteristic plateau is typically located between a higher variance of the estimator for smaller thresholds and a larger bias of the estimator for bigger thresholds. The estimate of the lower TDC and the corresponding threshold k is chosen according to the latter plateau.

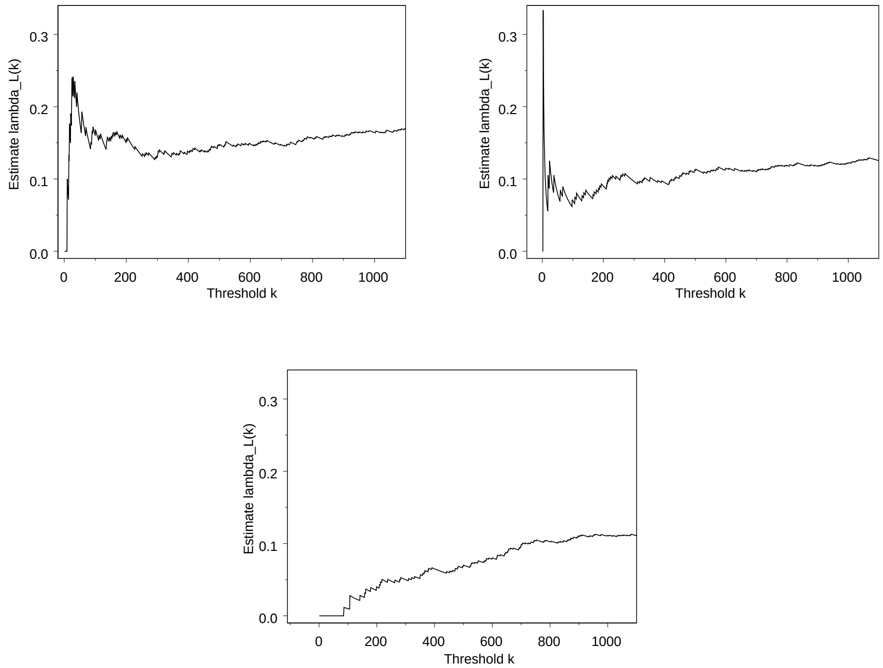

Fig. 7. Estimates $\hat{\lambda}_{L,n}(k)$ of the lower tail-dependence coefficient for various thresholds k for returns R_i^j (upper left plot), volatility-weighted returns $\tilde{R}_i^j$ (upper right plot), and GARCH residuals of the volatility-weighted returns (lower plot) for GM and IBM over the years 1993-1998.

Figure 7 indicates that the original GM-IBM returns are lower-tail dependent with $\hat{\lambda}_{L,n} = 0.15$. The volatility weighted returns show less pronounced tail dependence with $\hat{\lambda}_{L,n} = 0.1$. Finally, the GARCH residuals of the volatility weighted returns are lower-tail independent according to the absence of any plateau; see the lower plot in Figure 7. However, the original returns and the volatility weighted returns are by no means i.i.d. Therefore the question is: Are the characteristic plateaus induced by the various temporal dependence structures of the data? For example, [BDE03] stop after the volatility weighting and draw several conclusions about the distributional dependence, although their deseasonalized high-frequency data set still shows a pronounced volatility clustering. In a forthcoming paper, we will dig into the question how much tail dependence can be introduced into a tail independent data set by applying certain transformation (which cause temporal dependence).

In contrast, we point out that the correlation coefficient is not significantly different for all three data series. The original GM-IBM returns have a correlation coefficient of 0.24, the volatility weighted returns possess a correlation coefficient of 0.23, and the GARCH residuals of the volatility weighted returns end up with a correlation coefficient of 0.22. This again unmistakably shows that the interpretational power of distributional dependence measures/models (such as copulas, Kendall's tau or tail dependence) has to be handled very carefully if the analyzed data are not i.i.d.

5 Conclusion

In this paper, we have surveyed and advocated the usage of copulas with a particular view towards financial applications. The recently developed concepts of tail dependence and tail copulas are presented and some new results on statistical inference are stated. The assumption of i.i.d. data, which is necessary in order to obtain the latter results, turns out to be difficult to obtain for real financial time series. In fact, we illustrate for the GM-IBM high-frequency data set that the distributional dependence is very sensitive towards common filtering methods such as GARCH filtering. We conclude that the analysis of the distributional dependence of multidimensional financial data with temporal dependence is a rich and promising area, in which much remains to be done.

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