Stochastic Processes: Theory and Methods. Handbook of Statistics **19** (ed. C. R. Rao & D. N. Shanbhag) 171-213, Elsevier, 2001)

RANDOM WALK AND FLUCTUATION THEORY

N. H. BINGHAM

CONTENTS

PART I: RANDOM WALK ON EUCLIDEAN SPACE

§1. Introduction.

The theory and applications of random walks are ubiquitous in the modern probability literature, and random walks form perhaps the simplest and most important examples of *stochastic processes* - random phenomena unfolding with time. The term 'random walk' can be traced back (at least) to Pólya (1921) ('*zufällige Irrfahrt*' in the German), but the context is much older. If we think of $X_1, X_2, \ldots$ as the gains or losses of a gambler on successive plays of a gambling game, and the partial sums

$$S_n := X_1 + \cdots + X_n \qquad (S_0 := 0)$$

as his cumulative gain or loss to date (by time n), then the behaviour of the stochastic process $(S_n)_{n=0}^{\infty}$ - the *random walk* with *steps* X_n - describes the evolution of the gambler's fortune. In the gambling context, analysis of such aspects as the duration of play (when the player has finite capital), probability of eventual ruin etc., goes back to Pascal and Huyghens in the 17th century. For a detailed account of such work, see the classic 19th century history Todhunter (1949); for a modern treatment, see the classic text of Feller (1968), XIV.

§2. Simple random walk on $\mathbb{Z}$. The simplest non-trivial case is to let $X_1, X_2, \cdots$ represent the outcomes of a succession of independent tosses of a fair coin. Suppose the gambler bets on heads, and gains $+1$ for each head and (by symmetry) loses one on each tail. Then S_n represents his cumulative gain or loss on the first n plays; $(S_n)_{n=0}^{\infty}$ is called *simple random walk* (on the lattice, or on $\mathbb{Z}$). Even this deceptively simple process contains many surprises if its behaviour is analysed in detail. Results include:
1. The walk returns to its starting position at time $2n$ with probability

$$u_{2n} := P(S_{2n} = 0) = \frac{1}{2^{2n}} \binom{2n}{n}.$$

One has

$$u_{2n} \sim \frac{1}{\sqrt{\pi n}} \qquad (n \to \infty),$$

by Stirling's formula. The generating function is

$$U(s) := \sum_{k=0}^{\infty} u_k s^k = \sum_{n=0}^{\infty} u_{2n} s^{2n} = 1/(1-s^2)^{\frac{1}{2}},$$

since $u_{2n} = (-1)^n \binom{-\frac{1}{2}}{n}$ (Feller (1968), XI.3(b)).
2. The walk eventually reaches each integer with certainty (and so, reaches it infinitely often). In particular, if $S_0 := 0$ and

$$T := \inf\{n : S_n = +1\},$$

then

$$P(T < \infty) = 1.$$

Thus a gambler whose strategy is to play till first ahead and then quit is certain to make (and keep) an eventual profit.
3. With T as above,

$$ET = +\infty.$$

That is, the gambler above has infinite expected playing time before he realises his eventual terminal profit. In particular, the above strategy is unrealisable in practice, as it needs unlimited playing capital, and playing time, to deliver a profit with certainty.
4. The distribution of T is

$$P(T = 2n - 1) = (-1)^{n-1} \binom{\frac{1}{2}}{n},$$

and its generating function is

$$P(s) := E(s^T) := \sum_{k=0}^{\infty} s^k P(T = k) = (1 - \sqrt{1 - s^2})/s$$

(Feller (1968), XI.3; thus $P'(1-) = +\infty$ yields $ET = +\infty$ as above).
5. The distribution of the time spent positive (suitably defined) up to time $2n$ is given by

$$P(\sum_{r=0}^{2n} I(S_r > 0, \text{ or } S_r = 0 \ \& \ S_{r-1} > 0) = k) = \binom{2k}{k}\binom{2n-2k}{n-k}/2^{2n} \quad (k = 0, 1, .., n)$$

(the Chung-Feller theorem: Chung & Feller (1949), Feller (1968), III.4). This distribution is called the *discrete arc-sine law* (see below).
6. In consequence, the limiting distribution function of the fraction of time spent positive is

$$(2/\pi) \arcsin \sqrt{x} \qquad (0 \le x \le 1)$$

(the *arc-sine law*: Feller (1968), III.4). This has density

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}} \qquad (0 < x < 1).$$

This density is U-shaped - unbounded near $x = 0$ and $x = 1$, with its *minimum* at the central value $x = 1/2$. The interpretation is that a typical coin-tossing sequence is much more likely to be unbalanced - with one player ahead most of the time - than balanced, a result usually regarded as counter-intuitive when first encountered.
7. The number of paths (k, S_k) from $(0, a)$ to (n, b) which touch or cross the x-axis is the number of paths from $(0, -a)$ to (n, b). This is the *reflection principle*, proved by regarding the axis as a mirror and using symmetry (see Feller (1968), III.1, Grimmett & Stirzaker (1992), §5.3). This is the probabilistic version of Kelvin's *method of images* in electrostatics. It gives as a corollary the *ballot theorem* (III.4 below; Feller (1968), III.1, Grimmett & Stirzaker (1992), §5.11.7). For applications to barrier options in mathematical finance, see e.g. Bingham & Kiesel (1998), §6.3.2.

This basic process - simple random walk on $\mathbb{Z}^1$ - is so simple to describe that it may seem to lack depth. On the contrary: the extreme simplicity of structure means that very detailed questions concerning it may be analysed in extreme depth. Such questions include the following.
(i) Long runs of heads (or tails) and their analysis.
(ii) Range - number of points visited by time n.
(iii) Local time - time spent at the origin (or a point x) by time n, etc.
For an extensive recent treatment of simple random walk in depth, see Révész(1990), Part I. A similarly detailed treatment of simple random walk in $\mathbb{Z}^d$ is given in Révész (1990), Part II.

§3. Recurrence and transience.

The basic dichotomy in random walk concerns whether or not eventual return to the starting-point is certain. If so, the walk is called *recurrent*; if not, *transient*. For recurrent random walks, return to the origin *infinitely often* (i.o.) is also certain (because $\lim_{n\to\infty} 1^n = \lim 1 = 1$); for transient random walks, this event has probability zero (because for $p \in [0, 1)$, $\lim_{n\to\infty} p^n = 0$). Thus the total occupation time for the starting point - and similarly, for all other points visited - for a recurrent random walk is infinite, while for a transient random walk it is finite. As the total number of time-points $n = 0, 1, 2, \cdots$ is infinite, a transient random walk must necessarily have an infinite state-space.

If we write u_n for the probability of return to the starting-point at time n, f_n for the probability of *first* return to the starting-point at time n ($f_0 := 0$), one has the convolution relation

$$u_0 = 1, \qquad u_n = \sum_{k=0}^{n} f_k u_{n-k} \qquad (n \geq 1).$$

Forming the generating functions

$$U(s) := \sum_{n=0}^{\infty} u_n s^n, \qquad F(s) := \sum_{n=0}^{\infty} f_n s^n,$$

this becomes

$$U(s) = 1 + U(s)F(s),$$

giving the *Feller relation*

$$U(s) = 1/(1 - F(s)).$$

Write

$$f := \sum_{n=0}^{\infty} f_n;$$

then f is the probability of *eventual* return to the starting-point. So $f < 1$ for transience, $f = 1$ for recurrence. Thus one has recurrence if $u := U(1) = \sum u_n$ diverges, transience if $\sum u_n$ converges.

§4. Simple random walk on $\mathbb{Z}^d$; Pólya's theorem.

Suppose now we are in d-space $\mathbb{R}^d$, more particularly in the integer lattice $\mathbb{Z}^d$. We start at the origin, $(0, 0, \cdots, 0)$, and move to each of the $2d$ 'neighbouring' or 'adjacent' points - those with one coordinate ± 1 and the rest zero - with equal probability $1/(2d)$; successive steps are independent, each with this distribution. The result is called *simple random walk* in d dimensions.

It was observed by Pólya (1921) that recurrence or transience depends on the dimension d. For $d = 1$, we have

$$u_{2n+1} = 0, \qquad u_{2n} \sim 1/\sqrt{\pi n},$$

so $u = \sum u_n$ diverges: simple random walk in one dimension is recurrent. For $d = 2$, for return to the origin at time $2n$ one must have equal numbers - k, say - of positive and negative steps in the first coordinate, and equal numbers - then $n - k$ - in the second coordinate. Thus

$$\begin{aligned} u_{2n} &= \frac{1}{4^{2n}} \sum_{k=0}^{n} \frac{(2n)!}{k!k!(n-k)!(n-k)!} \\ &= \frac{1}{4^{2n}} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \\ &= \frac{1}{4^{2n}} \binom{2n}{n}^2 \\ &\sim 1/(\pi n) \qquad (n \to \infty), \end{aligned}$$

by Stirling's formula. Thus $u = \sum u_n$ diverges, and simple random walk in two dimensions is recurrent also.

For three dimensions, we have similarly

$$u_{2n} = \frac{1}{6^{2n}} \sum\nolimits_{j,k} \frac{(2n)!}{j!k!(n-j-k)!j!k!(n-j-k)!},$$

the summation being over all j, k with $j + k \leq n$. Then

$$u_{2n} = \frac{1}{2^{2n}} \binom{2n}{n} \sum\nolimits_{j,k} \Big(\frac{1}{3^n} \frac{n!}{j!k!(n-j-k)!} \Big)^2.$$

The terms in the large brackets are those of a trinomial distribution, which sum to one. So the sum is majorised by the maximal term, which is attained for both j and k near (within ± 1 of) $n/3$. Stirling's formula now shows that the sum is $O(1/n)$. As above, the term outside the summation is $O(1/\sqrt{n})$, so $u_{2n} = O(1/n^{3/2})$. Thus $u = \sum u_n < \infty$: simple random walk in three dimensions is transient. The same argument applied in d dimensions, using the d-variate multinomial distribution, gives the sum as $O(1/n^{\frac{1}{2}(d-1)})$, and so $\sum u_n$ converges as before. This proves:

Pólya's Theorem. *Simple random walk in d dimensions is recurrent for $d = 1, 2$, transient for $d = 3, 4, \cdots$.*

Geometrically, the result may be interpreted as saying that there is 'more room' - more ways to avoid returning to the origin - in higher dimensions.

The probability p of eventual return to the origin in three dimensions - $p < 1$ by transience - has been calculated; its numerical value is

$$p = 0.340537329544 \cdots.$$

For details and references, see Doyle & Snell (1984), §7.5, or Spitzer (1964), Ch. II Problems 10, 11.

§5. Random walks on $\mathbb{R}^d$.

Consider first the case of random walk on $\mathbb{Z}^d$. Starting at the origin, each d-tuple $\mathbf{k} := (k_1, \cdots, k_d)$ is reached in one step with probability $\mu_{\mathbf{k}}$, where the $\mu_{\mathbf{k}}$, being probabilities, sum to one. Then if $\mu = (\mu_{\mathbf{k}})$ is the probability distribution for each step, that for n steps is the convolution power μ^n (or μ^{*n}, in the alternative notation), defined

inductively by $\mu^n := \mu * \mu^{n-1}$ ($\mu_0 := \delta_0$, the Dirac mass at the origin). If the starting position is x, the distribution after n steps is $\delta_x * \mu^n$. The transition probabilities

$$p_n(x,y) := P(S_n = y | S_0 = x)$$

are translation-invariant:

$$p_n(x,y) = p_n(y-x) := p_n(y-x,0).$$

Thus the transition probabilities are homogeneous with respect to the additive group structure of $\mathbb{Z}^d$ (we return to group-theoretic aspects in Part II below). The classic monograph of Spitzer (1964) deliberately restricts itself to this context, where the probabilistic structure is as unencumbered by any other as possible.

One may discard the discreteness of $\mathbb{Z}^d$, and work instead with $\mathbb{R}^d$. Here, since $\mathbb{R}^d$ is uncountable, measure-theoretic restrictions arise, and probabilities need to be calculated by integration rather than summation. For background, see e.g. Chung (1974), Ch. 8, Ornstein (1969). Again, one works with a sequence of partial sums $S_n := \sum_{k=1}^n X_k$ ($S_0 := 0$) of independent X_i with distribution μ, the distribution after n steps with starting-point x is $\delta_x * \mu^n$, and the additive group structure of $\mathbb{R}^d$ plays - via the addition in the partial sums - a dominant role.

Often the essential feature is that the distribution evolves through time $n = 1, 2, \cdots$ via the powers P^n of a matrix P, the transition probability matrix on some countable set S. Here we are in the context of *Markov chains* on state-space S. For a classical monograph treatment see Chung (1967); for a more recent account see Norris (1997). If S has a graph structure and the *nearest-neighbour condition* holds - that is, if a transition is possible from x to y then there is an edge from x to y - one speaks of a *random walk on the graph* G; see II.2 below. Of course, one can view any Markov chain in this way: one typically draws in a graph structure when classifying the states of a Markov chain, for instance. It is really a question of emphasis: when the properties - algebraic, geometric, topological - of the set S are themselves of interest, it is customary and convenient to use the language of random walks on S, and to take a dynamic viewpoint. If however the states $s \in S$ are not of particular interest in themselves, S serves merely as an index set to label the states, and one speaks of a Markov chain on S.

In both the random-walk and Markov-chain context, questions arise as to the nature - discrete or continuous - of both the time-set and the state-space. The traditional

usage has been to speak of Markov chains when time is discrete and Markov processes when time is continuous. However, one can argue that it is the nature of the state-space S which is the more decisive, and speak of Markov chains and processes according as to whether S is discrete or continuous; this is the point of view of the excellent text of Revuz (1984).

§**6. Harmonic analysis.** For a random walk $S_n = X_1 + \cdots + X_n$, with the step-lengths X_i independent with distribution μ, the basic operation is forming the nth convolution powers, as above:

$$\mu, \qquad \mu^2 := \mu * \mu, \qquad \mu^n := \mu * \mu^{n-1}, \cdots;$$

we will write μ^n as μ^{n*} when the convolution in the power needs emphasis. The operation of convolution involves an integration, and it is convenient to replace this by the simpler operation of multiplication. One does this by passing to the *characteristic function* (c.f.) or *Fourier-Stieltjes transform* of the X_j, or of μ:

$$\phi(t) := E(\exp\{itX_j\}) = \int e^{itx} \mu(dx) \qquad (t \in \mathbb{R}).$$

By the Uniqueness Theorem for characteristic functions, no information is lost thereby; by the Multiplication Theorem, the c.f. of an independent sum is the product of the c.fs. Thus the c.f. of S_n is ϕ^n, the nth power of the c.f. ϕ of each step-length.

The basic transience/recurrence dichotomy for random walks may be expressed in terms of the c.f. $\phi(t)$: the walk is transient if and only if $\Re 1/(1-\phi)$ is integrable in some neighbourhood of the origin. This is the *Chung-Fuchs criterion* (Chung & Fuchs (1951), as refined by Ornstein (1969)). This extends to locally compact abelian groups (see Part II below): with $\hat{\mu}$ the Fourier transform of μ, the random walk with step-length distribution μ is transient iff $\Re 1/(1-\hat{\mu})$ is integrable in some neighbourhood of the group identity (Kesten & Spitzer (1965)). Fourier analysis, like convolution, extends to the general setting of random walk on groups (again, see Part II).

We note in passing the extension of Pólya's theorem to general random walks on $\mathbb{R}^d$, due to Chung & Fuchs (1951).
(i) If $d = 1$ and the mean $E|X|$ of the step-length exists, the random walk is recurrent if $EX = 0$, transient otherwise.
(ii) If $d = 2$ and the variance, or $E(|X|^2)$, exists, the random walk is again recurrent if $EX = 0$, transient otherwise.
(iii) If $d \geq 3$, all properly d-dimensional random walks are transient.

§**7. Potential theory.** Potential theory as part of the classical theory of electricity and magnetism (and of gravitational attraction) grew out of the work of Green and Gauss in the 19th century. The theory of Brownian motion, or of the Wiener process, is a 20th century development; potential theory was linked to Brownian motion by Kakutani (1944). Classical concepts from the electromagnetic theory of continuous bodies such as equilibrium charge distribution, electrostatic capacity, potential and energy may be interpreted in terms of Brownian motion - the equilibrium charge distribution on the boundary ∂D of a conducting body D, for instance, is expressible in terms of the first hitting distribution of Brownian motion on ∂D. A succinct account emphasising the historical aspects is given by Chung (1995); for a textbook account, see e.g. Port & Stone (1978).

It was realised in the 1950s, through the work of Doob, Hunt and others, that one can develop a 'potential theory' for Markov processes, Brownian motion being distinguished by having as its potential theory exactly the classical one. Since random walks are particularly simple and important Markov processes, their potential theory has been particularly fully developed accordingly. The theory is seen in its barest essentials in the simplest possible context, random walk in $\mathbb{Z}^d$; the potential theory of such random walks is developed in detail in the classic book of Spitzer (1964).

The way in which the language of classical potential theory may be fruitfully generalised is illustrated by the concept of a harmonic function. Classically, a function f is harmonic if it satisfies Laplace's equation $\Delta f = 0$. This is a linear second-order elliptic partial differential equation; it may be discretised on a lattice - in the plane, say - as

$$\frac{1}{4}[f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)] - f(x,y) = 0.$$

With $p(x,y)$ the transition kernel, one may write this more concisely as

$$\sum\nolimits_y p(x,y)f(y) = f(x), \qquad \text{or} \qquad Pf = f.$$

In this form, P may be generalised to the transition function of an ordinary random walk, not just simple random walk on $\mathbb{Z}^2$ as in the example above. One calls functions f *harmonic* if they satisfy $Pf = f$.

The basic transience-recurrence dichotomy depends on the existence of a *Green function*,

$$G(x,y) := \sum\nolimits_{n=0}^{\infty} p_n(x,y)$$

(or $\int_0^\infty p(t,x,y)dt$ in continuous time). Random walks for which a Green function *exists* - that is, for which this sum or integral converges - are *transient*; when the Green function

does not exist (is identically $+\infty$) the walk is *recurrent*. Transient potential theory involves study of the Green function $G(x,y)$ (as in Spitzer (1964), VI); recurrent potential theory involves instead the Green kernels $G_n(x,y) := \sum_{r=0}^{n} p_r(x,y)$ (as in Spitzer (1964), III, VII).

Discrete Laplacian. We saw above that $Pf = f$ serves as a discrete analogue of Laplace's equation $\Delta f = 0$. This motivates the definition of

$$\Delta := P - I$$

as the *discrete Laplacian*. See e.g. Woess (1994), §4B, Biggs (1997), §§9, 10 and the references cited there for background; we return to this in Part II below.

§**8. Coupling.** While the Fourier and potential-theoretic aspects of random-walk methodology are quite classical, the coupling method is more recent. Suppose that the probability measures of two stochastic processes (in discrete time, say) are to be compared. The coupling method is to construct both processes on the same probability space - with the given measures as their distributions - and seek to compare the measures by comparing the processes themselves directly - that is, *pathwise*. The method originates with Doeblin (1938), in the context of Markov chains. It was developed by Ornstein (1969) for random walks (and so appears in the second - 1976 - edition of Spitzer (1964), but not the first). A monograph treatment of coupling generally is given by Lindvall (1992); see especially II.3, III.2.12 there for random walks.

One of the most successful applications of coupling is to proving the convergence theorem for (ergodic) Markov chains: one starts two independent copies of the chain, one in an arbitrary (or the given) starting distribution, the other in the stationary distribution, and runs them till they meet. One can then consider them as having 'coalesced' (because of the Markov property), and the convergence theorem follows rapidly from this. Another success of the coupling method is its use in proving renewal theorems (I.9 below; cf. Lindvall (1992), II.1, III.1, V.5). Many of the results of Part II below on random walks in more general contexts - such as finite groups, for example - are naturally proved by coupling methods; see e.g. Aldous (1983), Diaconis (1988), 4E.

§**9. Renewal theory.** Consider first the classical setting involving replacement of components - lightbulbs, say. At time 0, a new lightbulb is fitted, and used non-stop until it fails, when it is replaced; this replacement bulb is used non-stop till failure and then replaced, etc. With X_i the lifetimes of successive bulbs, $S_n := \sum_{i=1}^{n} X_i$,

$$N_t := \max\{n : S_n \leq t\},$$

N_t is the number of failures up to and including time t. Then $N := (N_t : t \geq 0)$ is called the *renewal process*. The lightbulb in use at time t is the $(N_t + 1)$th (as $S_{N_t} \leq t < S_{N_t+1}$); the mean of this is the *renewal function*,

$$U(t) := E(N_t + 1) = \sum_{n=0}^{\infty} (n+1) P(N_t = n) = \sum_0^{\infty} F^{n*}(t),$$

with F the distribution function of the lightbulb lifetimes.

Here - as lifetimes are non-negative - the random walk (S_n) is concentrated on the half-line $[0, \infty)$. More generally, one may consider random walks (S_n) on the line; with P the probability measure of the step-length,

$$U := \sum_{n=0}^{\infty} P^{n*}$$

is the *renewal measure*. Its study - in particular, of its asymptotic properties - is called *renewal theory on the line*. Similarly in more general settings such as groups: renewal theory is the study of the asymptotics of the Green kernel $G := \sum_{n=0}^{\infty} P^{n*}$.

The basic result is Blackwell's renewal theorem (Blackwell (1953)): if F has mean $\mu \in (0, \infty]$ and is non-arithmetic (or non-lattice: the support of F is not an arithmetic progression),

$$U(t+h) - U(t) \to \begin{cases} h/\mu & (t \to +\infty), \\ 0 & (t \to -\infty). \end{cases}$$

(The arithmetic case is similar but simpler. This is the Erdös-Feller-Pollard theorem; see Feller (1968), XIII.3.) Many proofs are known: see e.g. Feller (1971), XI.1 (renewal equation and direct Riemann integrability), Lindvall (1977) (coupling), Bingham (1989) (Wiener Tauberian theory).

The renewal theorem extends to $\mathbb{R}^2$ (Chung (1952)): here the limit is 0 for all approaches to ∞. The same is true for $\mathbb{R}^d$ $(d \geq 2)$: see e.g. Nagaev (1979).

Renewal theory on groups. Random walks on groups and other algebraic structures are considered in Part II below; we pause here to discuss briefly renewal theory in such settings. The renewal theorem extends to locally compact abelian groups G (Port & Stone (1969)). Call G of *type II* if it is capable of supporting a random walk - with step-length law μ, say - whose renewal measure $\nu := \sum_{n=0}^{\infty} \mu^{n*}$ does not tend to zero at infinity, *type I* otherwise. Thus by the results above, $\mathbb{R}^d$ and $\mathbb{Z}^d$ are of type II for $d = 1$ (by the non-arithmetic and arithmetic cases of Blackwell's theorem), type I for $d \geq 2$. The general result for locally compact abelian groups G is that G is of type II iff G is isomorphic to $\mathbb{R} \oplus K$ or $\mathbb{Z} \oplus K$ with K compact.

The renewal theorem extends also to non-abelian groups. Recall that a group is *unimodular* if the left-invariant (Haar) measure is also right-invariant; it is *amenable* if it possesses an invariant mean, on the space of bounded uniformly continuous functions. The term amenable is understood in English to carry both its ordinary connotation and that of 'meanable' (French, *moyennable*, German, *mittelbar*); for background, see e.g. Greenleaf (1969), Eymard (1972), Bondar and Milnes (1981). The non-amenable case is always type I (Derriennic & Guivarc'h (1973)). In the unimodular case, type II groups are of the form $K \times E$ with K compact and E isomorphic to $\mathbb{R}$ or $\mathbb{Z}$ (Sunyach (1981)). The amenable, non-unimodular case is studied in detail by Elie (1982a) (especially p.260 and §§1.6, 3.14, 3.22, 5.1).

§**10. Limit theorems and Brownian motion.** Much of the core of classical probability theory is concerned with the limit theory of a sequence of partial sums S_n of random variables X_n (independent and identically distributed, in the simplest case). For example, the trilogy of classical limit theorems - the (strong) law of large numbers, the central limit theorem and the law of the iterated logarithm - concerns just this. Since S_n is a random walk, all of this is random-walk theory in some sense. For our purposes, however, we prefer to regard this material - the classical limit theorems and the central limit problem - as part of general probability theory, and refer to the excellent textbook treatments in the classic texts of Feller (1971) and Chung (1968). We focus here more on the aspects specific to random-walk theory - the recurrence/transience dichotomy, and the specifically stochastic-process aspects.

Of course, in limit theory one is concerned with S_n as $n \to \infty$. As the number n of steps increases the influence of each individual step decreases, and in the limit it is lost altogether. One thus expects the setting of a random walk to go over on passage to the limit to the setting of a stochastic process in continuous time and state-space, and this is indeed true. In the simplest case when the step-lengths have finite variance, the limiting process obtained is *Brownian motion* or the *Wiener process*. The mathematics necessary to handle the passage to the limit is the theory of weak convergence of probability measures, specifically the *Erdös-Kac-Donsker invariance principle*; for an excellent account, see Billingsley (1968). The continuous framework of Brownian motion or some other limiting process - a stable process, or a diffusion, for instance - lurks behind much of the discrete framework of random walks.

Instead of obtaining a continuous-time or continuous-state process from a random walk by a limiting procedure, one may instead begin in a continuous time and state-space

framework. In this setting, the analogue of a random walk is a *Lévy process* - a stochastic process with stationary independent increments. For a recent monograph account, see Bertoin (1996). This book is particularly noteworthy for its treatment of the potential-theoretic (Ch. II) and fluctuation-theoretic (Ch. VI) aspects; cf. I.7 above and Part III below.

§**11. Conditioned random walks.** It frequently happens that one needs to deal with a random walk - or other stochastic process - in which one wants to *condition* on some event or other having happened (or - more often - not having happened) by time n. This idea has been current at least since the work of Dwass & Karlin (1963). In limit theorems, one expects to obtain as limit process a process obtained from a familiar one by some conditioning operation. For example, the Brownian bridge is obtained from Brownian motion by conditioning on return to the origin at time $t = 1$ (Billingsley (1968)), the three-dimensional Bessel process is obtained by conditioning Brownian motion to stay positive, and further processes such as Brownian excursion, Brownian meander and their relatives are obtainable from Brownian motion by conditioning operations of various kinds. Asmussen (1982) gives conditioned limit theorems for random walks, with various applications to queues and risk theory (we defer consideration of such applied topics to Part III). For further results, background, and references, we refer to Bertoin & Doney (1994), (1996).

It may be that the event on which one is conditioning has small probability, vanishing in the limit. For example, one may have a random walk which drifts to $-\infty$, conditioned to stay positive. In such situations, one is visibly focussing on highly atypical behaviour, and the appropriate theory for handling such cases is that of *large deviations*, for background on which we refer to, e.g., Deuschel & Stroock (1989), Dembo & Zeitouni (1993). In the random-walk or risk-theoretic context, the basic technique is to pass to the *associated* random walk, a technique originating with Cramér. See e.g. Feller (1971), XII.4 for theory, Asmussen (1982) for applications.

PART II. RANDOM WALKS IN MORE GENERAL CONTEXTS

§1. Random walks and electrical networks. We have seen (I.7) that electromagnetism - in particular, electrostatics in continuous media - is relevant to random walks, via potential theory. The theory of current electricity in networks of wires is also relevant, an observation due to Nash-Williams (1959). This viewpoint has been given an excellent textbook treatment by Doyle & Snell (1984); their book is largely motivated by an attempt to 'explain' Pólya's theorem.

Suppose we have a network of conducting wires, joining nodes $x, y, \ldots$. Write $R_{x,y}$ for the resistance of the wire xy from node x to node y ($R_{x,y} := +\infty$ if there is no such edge). Note that $R_{x,y} = R_{y,x}$, which reflects the time-reversibility of the physics of steady electrical currents. Write

$$C_{x,y} := 1/R_{x,y}$$

for the conductance of the wire,

$$C_x := \sum\nolimits_y C_{x,y},$$

$$P_{x,y} := C_{x,y}/C_x.$$

Then one can define a stationary Markov transition matrix $P = (P_{x,y})$, and this is reversible, since

$$C_x P_{x,y} = C_{x,y} = C_{y,x} = C_y P_{y,x}$$

(see Kelly (1979) for background on reversibility, and II.2 below). If

$$C := \sum\nolimits_x C_x = \sum\nolimits_{x,y} C_{x,y} < +\infty,$$

the chain is ergodic with stationary distribution

$$\pi_x = C_x/C$$

($C < +\infty$ for finite networks; for a full treatment of infinite networks, we refer to Zemanian (1992)). Conversely, reversible chains can arise in this way from networks: reversibility characterises those ergodic chains that arise from electrical networks.

The link between random walks and electrical networks is developed in detail in the book by Doyle & Snell (1984). Key results include the following.
(i) Thomson's Principle (or Kelvin's Principle): the flow of electricity through a network minimises the energy dissipation.

(ii) Rayleigh's Monotonicity Law: increasing resistances can only increase effective resistance (that is, decrease currents).
Used in combination, one can combine short-circuiting (removal of resistance between nodes) and cutting (removal of connection between nodes) to introduce powerful comparison methods between electrical networks, and hence - and this is the point of the method - between the analogous random walks. For example, Doyle & Snell (1984), §5.9, exploit Rayleigh's method to give an electrical proof of Pólya's Theorem (short-circuiting shows recurrence in the plane, cutting shows transience in space). They also consider (Ch. 6) random walks on trees, to which we return in II.5 below.

The energy ideas of the electrical analogy have been used by T. Lyons (1983) to give a general recurrence/transience criterion for reversible Markov chains. His method has interesting connections with Riemann surfaces and Riemannian manifolds, and with Brownian motion on them; see II.4 below.

The electrical network method has recently been used by a number of authors to simplify and extend various results on random walks on graphs (see II.2 below) and networks. See for example Telcs (1989), Tetali (1991), Palacios (1993), (1994) and the references cited there.

§**2. Random walk on graphs.** A *graph* $G = (V, E)$ is a pair consisting of a set V of *vertices*, and a set E of *edges* joining some pairs of vertices. For each $v \in V$, we may consider the set N_x of *neighbours* of x - vertices y with an edge joining x to y. One may define *simple random walk* on the graph G by specifying that starting at x, the particle moves to each neighbour $y \in N_x$ with equal probability $1/|N_x|$, successive steps of the walk being independent. More general probability assignments are possible: here the step from x to $y \in N_x$ has probability $p_{x,y}$ (here $y = x$ is allowed, representing either a loop from x to itself or the possibility of staying put, and we can extend to all y by putting $p_{x,y} = 0$ if y is not a neighbour of x).

Under mild conditions (such as irreducibility - all states being accessible from each other - aperiodicity - absence of cycling - and recurrence - some, hence all, states recurrent rather than transient), there exists a *limiting*, or *stationary*, *distribution* $\pi = (\pi_x)_{x \in V}$. This has the property that if the system is started in π - so

$$P(S_0 = x) = \pi_x, \qquad (x \in V)$$

- it remains in π after one step:

$$\pi_y = \sum\nolimits_{x \in V} \pi_x p_{x,y} \qquad (y \in V),$$

hence by induction for any number of steps.

One may interpret $\pi_x p_{x,y}$ as the 'probability flux' from x to y. If the flux along this edge is the same in the reverse direction,

$$\pi_x p_{x,y} = \pi_y p_{y,x} \qquad (x, y \in V),$$

then the walk is called *reversible*: its stochastic behaviour is the same if the direction of *time* is reversed (this reversibility condition is also called the *detailed balance* condition). A monograph treatment of reversibility is given by Kelly (1979), in the more general context of Markov chains. As noted in II.1 above, reversible random walks on graphs are exactly those for which an interpretation in terms of electrical networks is available.

Randomised algorithms. The general area of random walks on graphs and reversible Markov chains has recently become highly topical. There is an extensive recent monograph treatment by Aldous & Fill (1998+). Part of the motivation comes from theoretical computer science, in particular the theory of *randomised algorithms*. In many situations, algorithms to perform certain complicated tasks numerically involve conditional statements ('if ... then ..., otherwise ...'), so that it is not clear in advance how many steps, or iterations, the program will take. An analysis of the computational complexity may deal with the worst case, but this is quite unrepresentative: it is usually better to focus on an average case, in some sense. The ability to analyse, and simulate efficiently, random walks on graphs representing the flow diagrams of - possibly very complicated - algorithms, and in particular their stationary distributions, is thus a valuable aid to assessing the computational complexity of many problems. Also, in many deterministic problems - approximate counting, volume estimation etc. - it is much more efficient to use a randomised algorithm rather than a deterministic one. Now, analysis of the computational complexity involves analysis of the convergence rates of the relevant random walks and Markov chains, for which an extensive theory is available. For the algorithmics background, see e.g. Sinclair (1993), Motwani & Raghavan (1995); for surveys of random walks on finite graphs motivated by this, see Lovász (1986), Lovász & Winkler (1995).

Algebraic aspects. For the purposes of probability theory on graphs, much useful information is provided by the subject of *algebraic graph theory*, for which see Biggs (1974), and the associated *algebraic potential theory* (Biggs (1997)). The algebraic potential theory hinges on the discrete Laplacian (cf. I.7), and the associated Dirichlet problem. From this point of view, the link between random walks and electrical networks is that both can be expressed as Dirichlet problems on a graph.

Boundary theory. For random walks on infinite graphs, much of the interest is on behaviour at infinity and boundary theory. For a detailed account of this important subject, see Woess (1991), (1994). Among the other methods relevant here, we mention *isoperimetric inequalities*. It would take us too far afield to go into details; we refer for background to Dodziuk (1984), Dodziuk & Kendall (1986), Gerl (1988), Woess (1994) §3A.

Spectral methods. The classical method for studying the rate of convergence of a random walk on a finite Markov chain (or, here, a finite graph) is to use spectral theory. If the transition matrix of the chain is P, the n-step transition matrix is P^n. If P is diagonalised by finding its eigenvalues λ_i (arranged in decreasing order: $1 = \lambda_1 \geq \lambda_2 \geq \ldots \geq -1$, where $\lambda_1 = 1$ by the Perron-Frobenius theorem) and eigenvectors, and we form the diagonal matrix Λ of the λ_i, then P^n can be read off in terms of Λ^n, and the convergence behaviour of P^n as $n \to \infty$ is determined by λ_2, the second largest eigenvalue. For recent accounts of this spectral approach to the rate of convergence of a Markov chain, see Diaconis & Stroock (1991) in the reversible case, Fill (1991) in the non-reversible case. Alternatives to the spectral approach are available when the Markov chain, or graph, has special structure, and we turn to such cases in the sections below.

Cover times. One problem of particular interest for random walks on graphs is that of *cover times* - the time it takes for the walk to visit every state. For background, see e.g. Aldous (1989), Broder & Karlin (1989), Kahn *et al.* (1989), Zuckerman (1989), Ball *et al.* (1997).

§3. Random walks on groups. If $x_1, x_2, \cdots$ are independently chosen from a distribution μ on a group G, then the sequence of products

$$s_n := x_n x_{n-1} \cdots x_2 x_1 \qquad (s_0 := e)$$

is called the *random walk* on G generated by μ. If G is abelian, it is customary and convenient to write the group operation additively:

$$s_n := x_1 + \cdots + x_n \qquad (s_0 := 0).$$

Probability theory on groups - in particular, the theory of random walks - and the closely related study of harmonic analysis on groups have been developed principally for the case of locally compact groups G, to which we confine ourselves here. The (locally compact) abelian - lca - case is treated in detail in Rudin (1962); the compact case - using the Peter-Weyl theory of group representations - in Vilenkin (1968), Ch. 1. For the general

locally compact case, see Heyer (1977). The link with random walk on graphs is given by the *Cayley graph* of a group (see e.g. Biggs (1974), Ch. 16).

Ehrenfest model. For purposes of illustration, we mention here a classical instance of a random walk on a group, of some historical and physical interest. Let $G = \mathbb{Z}_2^d$, the additive group of d-tuples of 0s and 1s modulo 2 (thus $1 + 1 = 0$). This is motivated by the famous Ehrenfest urn model of statistical mechanics, where d balls are distributed between two urns, labelled 0, 1 (so the space $\mathbb{Z}_2^d$ of d-tuples describes the occupancy states of the balls). A move, or step, consists of choosing a ball at random and transferring it to the other urn. This generates a random walk on $G = \mathbb{Z}_2^d$ (and also a nearest-neighbour random walk on the graph corresponding to $\mathbb{Z}_2^d$, the unit cube C_d in d dimensions). As the structure of G is so simple, the behaviour of the walk is straightforward to analyse (Kac (1959), III.11). Now the original way to analyse this model involves counting the number of balls present in one (or the other) urn - superficially simpler, as the number of states is thereby reduced from 2^d to $d+1$. In fact, the analysis is now more complicated (Kac (1959), III.7-10). It involves certain special functions - discrete orthogonal polynomials, the *Krawtchouk polynomials* - which arise in the harmonic analysis of the relevant Gelfand pair (III.4 below).
Note. In the statistical mechanics context, d is of the order of magnitude of Avogadro's number (c. 6×10^{23}), so 2^d - the number of states, and the recurrence time of the extreme states $(0, \cdots, 0)$ and $(1, \cdots, 1)$) - is so vast as to make the theoretical recurrence of states with such astronomically large recurrence times unobservable in practice. The importance of the Ehrenfest model is to reconcile the observed irreversibility of systems at macroscopic level with the reversibility of the dynamics describing them at microscopic level. This theme - the question of the 'arrow of time' - is of fundamental importance in physics. For background and references, see e.g. Bingham (1991), Bingham (1998), §1.11.

Card-shuffling. If a pack of n cards ($n = 52$ in the case of ordinary playing cards) is shuffled, the objective is to start from the initial distribution - which is highly patterned, reflecting following suit in the play of the previous hand - and end with a patternless or uniform distribution. There are $n!$ permutations of the cards (note that 52! is enormous! - c. 8.05×10^{67}). The usual method of shuffling - riffle shuffling - is analysed in detail by Bayer & Diaconis (1992), Diaconis, McGrath & Pitman (1995). Suppose distance between distributions is measured, as usual, by variation distance (or norm):

$$d(\mu, \nu) = \|\mu - \nu\| := \frac{1}{2}\sup_A\{|\mu(A) - \nu(A)|\}.$$

This exhibits the Aldous-Diaconis 'cut-off phenomenon': for μ a typical initial distribution, μ^k the distribution after k shuffles, π the uniform distribution - the limit distribution of

μ^k as $k \to \infty$ - $d(\mu^k, \pi)$ stays close to its maximum value 1 for k small, starts to decrease sharply around the 'cut-off' value $k \sim \frac{3}{2}\log_2 n$, and approaches zero geometrically fast for large k. For $n = 52$ as for actual playing cards, one may summarise this by saying that *seven shuffles suffice* to get close to uniform. For background, see e.g. Aldous & Diaconis (1986), Diaconis (1988), 3A.2, 4D. As well as a variety of probabilistic techniques, the mathematics involved is that of group representations and non-commutative Fourier analysis on finite groups.

Note. It is interesting to compare this line of work with a contrasting and more recent one due to Lovász & Winkler (1995). Here, one uses a randomised algorithm to achieve *exact* (rather than approximate) uniformity, but after a random number of steps. For the standard pack of $n = 52$ playing cards, the expected number of steps to achieve uniformity is c. 11.724.

Compact groups. Consider first the case of a compact group G - in particular, a finite group G. Here no question of behaviour 'at infinity' arises. Instead, the basic result here is the Itô-Kawata theorem: for a random walk with distribution μ on a compact group G, the convolution powers μ^n converge to (normalised) Haar measure on the closed subgroup G_μ generated by the support of μ (Heyer (1977), §2.1). We lose nothing for most purposes by restricting from G to G_μ (or, we may restrict to μ whose support generates G), when we obtain: *on a compact (in particular, finite) group, random walk converges to the uniform distribution.* Of course, as we have seen above with the finite case of card-shuffling, interest here focusses in great detail on *how fast* this convergence takes place (Diaconis (1988); Urban (1997)).

Boundary theory. For infinite, discrete groups, one of the main questions is to study how the random walk 'escapes to infinity'. Thus the behaviour of the group itself 'at infinity' - growth properties, etc. - is crucial. Furthermore, it is usually better to seek an appropriate compactification of the state-space - by adjoining a suitable boundary - so that the behaviour of the walk on this enlarged state-space is more informative, or better behaved, or both. Boundary theory is too vast a subject for us to do more here than point the reader to suitable references for a full account; see e.g. Furstenberg (1971), Kaimanovich & Vershik (1983), Varopoulos *et al.* (1992), Woess (1994), Sawyer (1997), Kaimanovich (1991). The general theme is that each of the structure of a group G, and the behaviour of random walk on G, is highly informative about the other. Similar remarks apply to the behaviour (especially at infinity) of random walks on graphs; see II.2 above and II.4 below. Particular kinds of group have been studied in greater depth; for the

boundary theory of random walks on Fuchsian groups, for example (groups G of Möbius transformations which act discontinuously on some G-invariant disc: see e.g. Beardon (1983), §6.2), see Series (1983).

Kesten's problem. The most basic question about random walk on groups is the recurrence/transience dichotomy: when is the random walk generated by μ on a group G recurrent or transient? (as above, it may be appropriate to restrict to $G_\mu = G$). If we recall the Chung-Fuchs criterion of I.6, we see that the special nature of $\mathbb{Z}^1$ - that it can support a recurrent random walk - is revealed by the symmetric probability laws μ (whose mean is zero), and that of $\mathbb{Z}^2$ by the symmetric laws with finite variance - in particular, with compact support. So if we restrict to probability laws μ which are symmetric (x and x^{-1} have the same distribution) and of compact support, $\mathbb{Z}^d$ can support a recurrent random walk if $d = 1$ or 2, but not otherwise. Groups G that can support a recurrent random walk generated by a symmetric μ of compact support are called *recurrent groups*; other groups are called *transient groups*. The question of which groups are recurrent and which are transient has become known as *Kesten's problem*, in honour of early work by Kesten (1959), (1967), Kesten & Spitzer (1965). Note that we already have the following examples of recurrent groups: finite groups; $\mathbb{Z}$, $\mathbb{Z}^2$.

It turns out that these examples are, in a sense, the prototypes for *finitely generated* recurrent groups: the only recurrent groups which are finitely generated are $\{e\}$, $\mathbb{Z}$, $\mathbb{Z}^2$ and finite extensions of them (Varopoulos: see Varopoulos *et al.* (1992), Ch. VI). The solution depends on the *volume growth* of a discrete, finitely generated group (defined in terms of the word metric - the graph metric of the Cayley graph: *ibid.*, VI.2, Woess (1994), §2C). Then finitely generated groups are recurrent iff their volume function $V(k)$ can have growth of at most $O(k^2)$ (*ibid.*, VI.6). The theory is extended to (locally compact) unimodular, compactly generated groups (prototype: $\mathbb{R}^d$) in Varopoulos *et al.* (1992), Ch. VII, and to unimodular Lie groups in Varopoulos *et al.* (1992), Ch. VIII (Kesten's problem for connected Lie groups is solved in Baldi (1981); for background on random walks on Lie groups, see Guivarc'h, Keane & Roynette (1977)).

§**4. Brownian motion on Riemannian manifolds.** The motivating problem for the Varopoulos theory on Kesten's problem for groups was the analogous question for certain Riemannian manifolds, in particular for covering manifolds M_1 of a compact manifold M (for background on covering manifolds, see e.g. Chavel (1993), Ch. 4). The deck transformation group Γ of the normal covering $P_1 : M_1 \to M$ is finitely generated and compact, since this holds for the fundamental group $\pi_1(M)$ of M. Varopoulos' theorem

states that Brownian motion on M_1 is recurrent if and only if the deck transformation group Γ is recurrent (Varopoulos *et al.* (1992), X.3), and we know from the above that this holds if and only if Γ is a finite extension of $\{e\}$, $\mathbb{Z}$ or $\mathbb{Z}^2$. For background to potential theory on manifolds and graphs (such as the Cayley graph of a finitely generated group, as here), see e.g. Ancona (1990), III, Lyons & Sullivan (1984) and the references cited there, and Biggs (1997). This method of studying manifolds via graphs is called *discretization*, since the graph serves as a discrete analogue of, or approximation to, the manifold.

Heat kernels. The heat kernel $p(t, x, y)$ - the fundamental solution of the heat equation

$$(\Delta + \partial/\partial t)u = 0,$$

with Δ the Laplacian of the manifold - plays a decisive role in the analysis and potential theory of a manifold, as well as the behaviour of Brownian motion there; for background see Davies (1989) and the references cited there. The discretization procedure above replaces this by its analogue on the graph, also called a heat kernel. This may be studied in continuous time (as on the manifold), or discrete time, when one obtains $p_n(x, y)$, the n-step transition probabilities of the random walk on the graph. Recent results on heat kernels on graphs are due to Pang (1993), Davies (1993); we shall return to this question later for graphs with special structure.

§5. Random walks on homogeneous spaces, Gelfand pairs, hypergroups and semigroups.

1. Homogeneous spaces. Random walks often occur in settings which are not themselves groups, but in which a group structure is nevertheless present. If G is a group and K a compact subgroup, G acts on the coset space $M := G/K$, which is a *homogeneous space*; random walk in such contexts have been surveyed by Elie (1982b), Schott (1984). With G, K suitable Lie groups, M may be given a Riemannian manifold structure; certain M arising in this way (those for which the curvature tensor is invariant under parallel translations) are *symmetric spaces* in the sense of Elie Cartan. For a monograph treatment of these, including Cartan's classification, see Helgason (1962). A particularly important case is that of the spheres Σ_k (a k-dimensional manifold of constant positive curvature in $(k+1)$-dimensional Euclidean space): these are compact symmetric spaces of rank one, given by

$$\Sigma_k = SO(k+1)/SO(k)$$

(Helgason (1962), X.3, Example III). For random walks on Σ_k, see Bingham (1972).

2. Gelfand pairs. Closely related to this is the concept of a *Gelfand pair*: pairs (G, K)

as above in which the convolution algebra of functions in $L_1(G)$ bi-invariant under K is commutative. For random walks in this context, see the survey of Heyer (1983). Prototypical examples of Gelfand pairs include $(G, K) = (SO(k+1), SO(k))$ above, relevant to random walk on spheres, in the infinite (or continuous) case, $(G, K) = (\mathbb{Z}_2^d \times S_d, S_d)$ (with S_d the symmetric group on d objects), giving the unit d-cube relevant to the Ehrenfest urn model in the finite case (Diaconis (1988), 3F, Remark 3). Gelfand pairs provide the machinery needed to lift a random walk on a Markov chain to a random walk on a group. For background, see Letac (1981), (1982), Diaconis (1988), 3F, 3G and the many references cited there.

3. Hypergroups. If δ_x denotes the Dirac measure at x for x in a group G, the convolution is given by $\delta_x * \delta_y = \delta_{x.y}$. The convolution $\delta_x * \delta_y$ can usefully be defined on some structures other than groups, called *hypergroups*, and here also one can study random walks. For full detail on this important subject, we refer to the survey of Heyer (1984) and the monograph of Bloom & Heyer (1994).

4. Semigroups. Here one has no inverse operation as in a group, only the product operation; typical examples are the reals under multiplication, and matrices under matrix multiplication. The resulting structures are of course less rich than their group-theoretic counterparts, but nevertheless a theory of random walks on semigroups - including, in particular, questions of recurrence and transience - has been developed in some detail. For background, see e.g. Högnäs (1974), Mukherjea and Tserpes (1976).

§6. Random walk on graphs with special structure.

1. Graphs with symmetry properties. The d-cube $\mathbb{Z}_2^d$ considered earlier in connection with the Ehrenfest model is a good example of a graph with a high degree of symmetry. Other examples include:

(i) the Platonic solids (classical regular polyhedra: tetrahedron, cube, octahedron, dodecahedron, icosahedron),

(ii) the Archimedean solids (semi-regular polyhedra) - prisms, antiprisms, and thirteen others, including the truncated icosahedron/soccer ball, which has recently achieved fame as the model for the C_{60} or buckminsterfullerene molecule (a new form of carbon),

(iii) higher dimensional polytopes (see e.g. Coxeter (1973), VII).

Properties of random walks on regular graphs, polyhedra and polytopes have been studied in depth in a series of works by Letac & Takács (1980a), (1980b), Takács (1981), (1984), (1986). For background on the implications of symmetry and regularity properties of graphs, see Biggs (1974), Part Three. Random walks on highly symmetrical graphs have been studied by Devroye & Sbihi (1990), and on edge-transitive graphs by Palacios

& Renum (1998). See also Belsley (1998) for rates of convergence.

2. Pre-fractals. Many of the classical examples of fractal sets are nested fractals, obtained by some recursive construction and exhibiting some self-similarity property. The *fractal* is obtained by a limiting procedure; the graph obtained by terminating the recursive construction after finitely many steps is a *pre-fractal*. An important example is the *Sierpinski gasket*, a fractal obtained by starting from an equilateral triangle and recursively removing the opposite-pointing triangle forming its 'middle quarter'. The corresponding pre-fractal, the Sierpinski pre-gasket or Sierpinski graph, is illustrated in Falconer (1985), Fig. 8.4 §8.4. Random walk on the Sierpinski graph is considered in detail by Grabner (1997), Grabner & Woess (1997). There are interesting near-constancy phenomena, and connections with branching processes; for background, see e.g. Biggins & Bingham (1991), (1993). Jones (1996) obtains bounds for heat kernels on the Sierpinski graph, which - because of the special structure - are better than those available for more general graphs (Pang (1993), Davies (1993)). See also Hattori *et al.* (1990), Hattori & Hattori (1991).

3. Trees. Trees - graphs without circuits - are simpler to handle than general graphs. Doyle & Snell (1984), Ch. 6 use random walk on trees and Rayleigh's comparison method to give a new proof - intended as an 'explanation' - of Pólya's theorem.

The most interesting case is that of an infinite tree. If we have a transient random walk on an infinite tree, attention focusses on how the walk 'escapes to infinity', hence on compactifications of the state-space. For background and references, see the survey of Woess (1991) and the papers on trees cited there.

The key parameter for random walk on an infinite but locally finite tree is the mean number of branches per vertex. This can be identified with the exponent of the Hausdorff dimension of the boundary (R. Lyons (1990)). One can introduce a one-parameter family of random walks on such trees, where the tendency to transience (escape to infinity) may be balanced by a greater probability of choosing the branch back towards the root. By varying the parameter, a phase transition is obtained. For a full account, see R. Lyons (1990), (1992), Lyons & Pemantle (1992), the monograph Lyons & Peres (1998+), and Takacs (1998).

4. Crystallographic lattices. The lattice $\mathbb{Z}^2$ gives a - recurrent - tiling of the plane by squares; one can also tile the plane by equilateral triangles or by hexagons. That random walk on the triangular and hexagonal lattices is also recurrent was shown by Rogers (1985).

Suppose one takes the square lattice $\mathbb{Z}^2$, and replaces each lattice point by a small square whose diagonals form part of the grid-lines of the lattice. The resulting tiling of the plane by small squares and larger octagons is familiar from patterns on wallpaper and linoleum (usually rotated through $\pi/4$ for aesthetic reasons). This lattice too is recurrent; Rogers (1985) shows this, and gives a general method in terms of additive functionals of Markov chains. Results of this type are also given by Soardi (1990).

One may apply the method to other familiar lattices, such as the crystallographic lattices in 2 and 3 dimensions. These may be classified: in 2 dimensions, there are 17 such 'wallpaper groups', most of them represented on the famous wall decorations of the Alhambra in Toledo (Schwarzenberger (1974)). In three dimensions, there are over two hundred (there are two ways to count them, depending on whether or not laevo and dextro forms are distinguished). For background on these and crystallographic classification in higher dimensions, see Schwarzenberger (1980).

§**7. Variants on random walk.** A number of variants and generalizations of random walk have been studied, for mathematical interest or to model aspects of natural phenomena.

1. Random walk in random environments. Considering random walk on $\mathbb{Z}$ (for simplicity), suppose that each integer n is chosen independently to be one of two types; with these choices made, $\mathbb{Z}$ is now a 'random environment'. Now suppose that a particle performs a random walk on $\mathbb{Z}$, but with different transition probabilities from sites of the two types. This is a random walk in a random environment (RWRE). The motivation comes partly from the physics of random (or disordered) media. For details and references, we refer to Révész (1990), Part III, Lyons & Pemantle (1992), Pemantle & Peres (1995).

2. Reinforced random walk. Suppose that a particle performs a (nearest-neighbour) random walk, not choosing all neighbours with the same probability but showing a preference for sites already visited. This model - called reinforced random walk - is motivated by the habits, and learning behaviour, of humans and animals: one deepens one's knowledge of the known environment by re-visiting it, occasionally extending it by forays into the unknown. For background, see e.g. Pemantle (1988), Davis (1989).

3. Self-avoiding random walk. Suppose that a random walk evolves, but that the walk is not allowed to revisit states previously visited. The resulting process, called a self-avoiding random walk, models the behaviour of polymers and the like. The model is difficult to analyse, as the excluded volume restriction makes the current evolution of the path strongly dependent on the entire history of the path to date. For detailed accounts, see e.g. Barber

& Ninham (1970), Madras & Slade (1993), Hughes (1995), Ch. 7. Self-avoiding walk on the Sierpinski graph and gasket has been considered by Hattori *et al.* (1990), Hattori & Hattori (1991), motivated by physical applications.

4. Branching random walk. Branching processes model the reproductive behaviour of biological organisms; random walks may be used to model the spatial diffusion of such organisms. The two may be combined: in the branching random walk, particles perform random walk for some lifetime distribution (exponential, say); on death, they are replaced by the next generation, as in the usual branching-process model, each of whom performs a new random walk independently, starting from its birthplace. The resulting model, an idealization of a biological population evolving in time and space, has been analysed in considerable detail; see e.g. Biggins (1995) and the references cited there. We shall deal in Part III with the Lindley equation in connection with queueing; for related results for branching random walks, see Biggins (1998).

5. Tree models in mathematical finance. In probability theory one generally writes random walks additively, when the relevant group is abelian. In mathematical finance, however, one naturally thinks in terms of financial *returns* - gains per unit of capital invested - and it is now more natural to work with multiplicative random walks. The analogue of a random walk on $\mathbb{Z}$ taking steps ± 1 with probabilities p, q is now a *binomial tree*. The model is due to Cox, Ross & Rubinstein (1979), who used it to derive the discrete Black-Scholes formula (of which the Black-Scholes formula itself is a limiting case). For details, see e.g. Bingham & Kiesel (1998), §4.5.

PART III. FLUCTUATION THEORY

§1. Spitzer's identity. If we think of a random walk (S_n) as modelling the capital of a player in a gambling game, for instance, the monetary interpretation means that the large (or small) values of S_n are of particular interest. For many purposes - theoretical and practical - it is useful to focus attention on these explicitly, by considering the sequence

$$M_n := \max\{0, S_1, \cdots, S_n\}$$

of maximal partial sums to date (or, dually, of

$$m_n := \min\{0, S_1, \cdots, S_n\}).$$

The study of (M_n) and related quantities is referred to as the *fluctuation theory* of the random walk (the terminology is that of Feller (1949), (1968), III).

Passing from $(0, S_1, \cdots, S_n)$ - equivalently, from $(X_1, \cdots, X_n)$ - to $(M_1, \cdots, M_n)$ effects a useful data reduction, as the sequence of maxima will typically contain fewer (perhaps many fewer) distinct elements. On the other hand, the random walk (S_n) is *Markovian*, while the maximum sequence (M_n) is *non-Markovian*, which makes it much harder to analyse. The key result is the following, which links the distributions of M_n with those of $S_n^+ := \max(0, S_n)$. For $0 < r < 1$, $\Re\lambda \leq 0$, one has

$$\sum_{n=0}^{\infty} r^n E \exp\{\lambda M_n\} = \exp\{\sum_{n=1}^{\infty} \frac{r^n}{n} E \exp\{\lambda S_n^+\}\}$$

(Spitzer (1956): the result is called *Spitzer's identity*).

Write

$$f(\lambda) := E \exp\{\lambda X_1\}$$

for the characteristic function of the step-length distribution of the random walk (of course, this is usually defined as $E \exp\{itX_1\}$ for t real, but if we intend to continue t to a complex variable, as we do, the i serves no purpose). Thus f is defined for $\Re\lambda = 0$, a line in the complex λ-plane, and may be continued into a strip (which may degenerate to the line $\Re\lambda = 0$ above), or a half-plane, or the whole plane. For $0 < r < 1$, write

$$\omega_r^+(\lambda) := \exp\{\sum_{n=1}^{\infty} \frac{r^n}{n} E\{\exp \lambda S_n\} I(S_n > 0)\},$$

$$\omega_r^-(\lambda) := \exp\{\sum_{n=1}^{\infty} \frac{r^n}{n} E\{\exp \lambda S_n\} I(S_n \leq 0)\}.$$

Then $\omega_r^+(\lambda)$ is defined at least for $\Re\lambda \leq 0$, $\omega_r^-(\lambda)$ at least for $\Re\lambda \geq 0$, and are analytic in the respective open half-planes . On the intersection $\Re\lambda = 0$ of the two closed half-planes, where both are defined, one has

$$\begin{aligned}\omega_r^+(\lambda)\omega_r^-(\lambda) &= \exp\{\sum_{n=1}^{\infty} \frac{r^n}{n} E\exp\{\lambda S_n\}\} \\ &= \exp\{\sum_{n=1}^{\infty} \frac{r^n (f(\lambda))^n}{n}\} \\ &= 1/(1 - rf(\lambda)).\end{aligned}$$

This idea of taking a function defined only on a line (or strip) in the complex plane, and expressing it as a product of two functions each defined in complementary half-planes intersecting in this line and analytic in their interiors is often useful, as it allows the powerful machinery of complex analysis to be brought to bear. It may be traced to the work of Wiener & Hopf (1931) (cf. Paley & Wiener (1934), IV), and is accordingly known as the *Wiener-Hopf method*; for a survey of Wiener-Hopf methods in probability, see Bingham (1980). The factors ω_r^+, ω_r^- are called the *right* and *left Wiener-Hopf factors* of the random walk.

We may re-write Spitzer's identity as

$$\sum_{n=0}^{\infty} r^n E\exp\{\lambda M_n\} = \omega_r^+(\lambda)\omega_r^-(0) = \exp\{\sum_{n=1}^{\infty} \frac{r^n}{n} E\exp\{\lambda S_n^+\}\},$$

and there is a bivariate extension

$$\sum_{n=0}^{\infty} r^n E\exp\{\lambda M_n + \mu(S_n - M_n)\} = \omega_r^+(\lambda)\omega_r^-(\mu) \qquad (0 < r < 1, \Re\lambda \leq 0, \Re\mu \geq 0),$$

also called Spitzer's identity or the *(first) factorization identity*. Both are due to Spitzer (1956); for an excellent textbook treatment, see Chung (1974), Ch. 8.

Order statistics. Spitzer's identity deals with the *maximum* - largest order statistic - of the partial sums. It extends to other order statistics: see Wendel (1960), de Smit (1973a).

Generalisations. Factorisation identitities of this type can be proved in much more general contexts - such as Markov chains, Markov additive processes, etc. - and there is now a considerable theory. For background and details, see e.g. Arjas and Speed (1973), Asmussen (1989), Kennedy (1998).

§**2. Ladder epochs and heights.** Particular interest attaches to those partial sums which are maximal - those members of the sequence (S_n) which belong to the sequence (M_n) also. The zeroth partial sum is $S_0 := 0$; the first positive partial sum, Z say, is called

the first *strict ascending ladder height*; the first time T that this level is attained is called the first strict *ascending ladder epoch*. Thus T, Z are the time and place of first passage of the random walk to $(0, \infty)$. Considering the first non-negative partial sum S_n with $n \geq 1$ gives the *weak ascending* ladder height and epoch (first passage to $[0, \infty)$), and similarly for strong and weak descending ladder heights and epochs. Subsequent ladder epochs and heights may be defined by starting the process afresh at the first ladder epoch. The *ladder steps* are the gaps between successive ladder heights.

Of course, ladder variables may be *defective*: if the walk never enters $(0, \infty)$, one defines T, Z to be $+\infty$, and similarly for the other types of ladder variable. To proceed, one needs to classify by defectiveness or otherwise of the ladder variables. Discarding the trivial degeneracy of the step-length distribution F being concentrated at zero, one has exactly one of the following three alternatives (Feller (1971), XII, XVIII):
(i) *drift* to $+\infty$: $S_n \to +\infty$ a.s. (so $m := \min\{S_n : n \geq 0\} > -\infty$ a.s.),
(ii) *drift* to $-\infty$: $S_n \to -\infty$ a.s. (so $M := \max\{S_n : n \geq 0\} < +\infty$ a.s.),
(iii) *oscillation*: $\limsup S_n = +\infty$, $\liminf S_n = -\infty$ a.s. (so $m = -\infty$, $M = +\infty$ a.s.).
This drift/oscillation trichotomy is decided by

$$A := \sum_{n=1}^{\infty} \frac{1}{n} P(S_n > 0), \qquad B := \sum_{n=1}^{\infty} \frac{1}{n} P(S_n < 0) :$$

one has
(i) drift to $+\infty$ iff $A = \infty$, $B < \infty$,
(ii) drift to $-\infty$ iff $A < \infty$, $B = \infty$,
(iii) oscillation iff $A = B = \infty$
(in fact $\sum_{n=1}^{\infty} \frac{1}{n} P(S_n = 0) < \infty$ always, so one could use $P(S_n \leq 0)$, $P(S_n \geq 0)$ instead here). [Of course, if the mean step-length μ exists, the strong law of large numbers shows that we have drift to $+\infty$ if $\mu > 0$, drift to $-\infty$ if $\mu < 0$ and oscillation if $\mu = 0$, but matters are not so simple if μ is not defined.]

Write

$$L_n := \min\{k : k = 0, 1, \cdots, n : S_k = M_n\}$$

for the *first* time up to time n that the *maximum* it attained,

$$L'_n := \max\{k : k = 0, 1, \cdots, n : S_k = m_n\}$$

for the *last* time that the minimum is attained. Then L_n is the *last* occurrence of a strict ascending ladder-point, and dually L'_n is the last occurrence of a weak descending one.

Write T', Z' for the first weak descending ladder epoch and height. To solve the first-passage problem into the positive half-line $(0, \infty)$, one requires the joint law of (T, Z). It turns out that this is expressible in terms of the Wiener-Hopf factor ω^+, and that of (T', Z') in terms of ω^-. One has the *Spitzer-Baxter identity* (or *second factorization identity*):

$$\omega_r^+(\lambda) = 1/(1 - E(r^T \exp\{\lambda Z\})) = \sum_{n=0}^{\infty} r^n \int_{\{L_n = n\}} \exp\{\lambda S_n\} dP$$
$$(0 < r < 1, \quad \Re\lambda \leq 0)$$

and its dual form

$$\omega_r^-(\lambda) = 1/(1 - E(r^{T'} \exp\{\mu Z'\})) = \sum_{n=0}^{\infty} r^n \int_{\{L_n = 0\}} \exp\{\mu S_n\} dP$$
$$(0 < r < 1, \quad \Re\mu \geq 0)$$

(Spitzer (1960), Baxter (1958); cf. Port (1963), Chung (1974), Ch. 8).

The number of positive partial sums

$$N_n := \sum_{k=1}^{n} I(S_k > 0)$$

(occupation-time of the half-line $(0, \infty)$) is often important. For each n, the distributions of N_n and L_n coincide. Indeed, the laws of (N_n, S_n), (L_n, S_n) and $(n - L'_n, S_n)$ coincide - and similarly with S_n replaced by any function of $(X_1, \cdots, X_n)$ invariant under permutations of the X_i. This important result is called *(E. Sparre) Andersen's Equivalence Principle* (Andersen (1953/54); Chung (1974), Ch. 8).

Extremal factorization. The equivalence principle has many useful consequences. For example,

$$P(L_n = k) = P(L_k = k)P(L_{n-k} = 0)$$

follows easily from the Markov property of the random walk. This translates into

$$P(N_n = k) = P(N_k = k)P(N_{n-k} = 0),$$

an important but non-obvious property called *extremal factorization* (Port (1963), Heyde (1969)).

Note. The (ascending) ladder heights and epochs are also the (upper) records and record times of the partial-sum process. The term 'record' in the statistical literature usually denotes a record of the *readings* X_n rather than their partial sums. We shall use such

records in III.9 below; for background, see e.g. Foster & Stuart (1954), Bingham *et al.* (1987), §8.14.

§**3. Spitzer's arc-sine law.** Recall (I.1) the Chung-Feller theorem, giving the exact distribution of the time spent positive in simple random walk on $\mathbb{Z}$ (the discrete arc-sine law), and its limit distribution, the (continuous) arc-sine law. It turns out that the results above allow a definitive generalization of this result.

First, we note that the discrete arc-sine law for the occupation-time of a random walk holds, not only for the case of simple random walk on $\mathbb{Z}$ (where, recall, we had to take care over what we meant by the walk being 'positive'), but also whenever the step-length distribution F is symmetric and continuous. For this, see Feller (1971), XII.8. Note that this result has the remarkable property of being *distribution-free*: it does not depend on F, provided only that F is continuous and symmetric (see III.8 below).

For $0 < \rho < 1$, consider the probability distribution G_ρ on $[0,1]$ with density

$$g_\rho(x) := \frac{\sin \pi\rho}{\pi} x^{\rho-1}(1-x)^{-\rho}$$

(to see that this is a probability density, use $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$ and Euler's beta-integral). For $\rho = 0$, the density is singular at $x = 0$; consideration of Laplace transforms shows that

$$G_\rho \to \delta_0 \qquad (\rho \downarrow 0)$$

(weak convergence to the Dirac law at zero); similarly

$$G_\rho \to \delta_1 \qquad (\rho \uparrow 1).$$

Defining $G_0 := \delta_0$ and $G_1 := \delta_1$, one thus has a family of laws $\{G_\rho : 0 \leq \rho \leq 1\}$ on $[0,1]$ - the *generalized arc-sine laws* with *parameter* ρ (some authors use the alternative parametrization $1-\rho$), or with *mean* ρ: if X has law G_ρ,

$$EX = \rho$$

and its kth moment is given by

$$E(X^k) = \binom{k+\rho-1}{k} = (-)^k \binom{-\rho}{k} \qquad (k = 0, 1, \cdots).$$

For proof and background, see Dynkin (1961), Lamperti (1962), or e.g. Bingham *et al.* (1987), §8.6.2.

We may now formulate *Spitzer's arc-sine law* (Spitzer (1956)): the fraction of time N_n/n that the random walk spends positive up to time n has a limit distribution as $n \to \infty$ iff

$$\frac{1}{n}\sum_{k=1}^{n} P(S_k > 0) \to \rho \in [0,1] \qquad (n \to \infty);$$

then the limit law is the generalized arc-sine law G_ρ, and these are the only possible limit laws. The condition above is called *Spitzer's condition*; it was proved recently by Doney (1995) that this is equivalent to the apparently stronger condition

$$P(S_n > 0) \to \rho \qquad (n \to \infty).$$

Thus the following conditions are equivalent.
(i) Convergence of probabilities: $P(S_n > 0) \to \rho$.
(ii) Convergence of means: $\frac{1}{n}\sum_{k=1}^{n} P(S_k > 0) \to \rho = \int_0^1 x dG_\rho(x)$.
(iii) Convergence of moments: $E[(N_n/n)^k] \to (-)^k \binom{-\rho}{k} = \int_0^1 x^k dG_\rho(x)$ $(k = 0, 1, \cdots)$.
(iv) Convergence in distribution: $N_n/n \to G_\rho$ in distribution.
Furthermore, these are the only possible limit distributions. One can extend the list above to include an even stronger statement - weak convergence of the Markov processes measuring the time-lapse since the last ladder epoch (Bingham (1973)).

Spitzer's condition holds when the random walk belongs to the domain of attraction (without centring) of some stable law, H say: if

$$S_n/a_n \to H \qquad (n \to \infty),$$

then

$$P(S_n > 0) = P(S_n/a_n > 0) \to 1 - H(0) = P(Y > 0),$$

where Y is a random variable with the stable law H. If Y (or H) has index $\alpha \in (0, 2]$ and skewness parameter $\beta \in [-1, 1]$, one has

$$\rho = \frac{1}{2} - \frac{1}{\pi\alpha} \arctan(\beta \tan \frac{1}{2}\pi\alpha)$$

(Zolotarev (1957); cf. Bingham *et al.* (1987), §8.9.2). There are partial results in the converse direction: Spitzer's condition implies a domain-of-attraction condition 'far from symmetry', but not 'close to symmetry'. For details and references, see Bingham *et al.* (1987), §8.9.2.

Spitzer's condition is equivalent to regular variation of the tail of the ladder *epoch* T (that is, for T to be in the domain of attraction of a one-sided stable law). The condition for regular variation of the tail of the ladder *height* Z is *Sinai's condition*:

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\lambda < S_n \leq \lambda x) \to \beta \log \lambda \qquad (x \to \infty) \qquad \forall \lambda > 1$$

(Sinai (1957); Bingham *et al.* (1987), §8.9.4).

§**4. Ballot theorems.** Suppose that two candidates, A and B, compete in a ballot, their finals scores being a and b votes respectively ($a > b$). The probability that the winning candidate A is ahead throughout the count is $a/(a+b)$. This classic result, the *ballot theorem*, stems from the work of Desiré André in 1887 (the reflection principle of I.2) and Whitworth's classic book *Choice and chance* (Whitworth (1886)). For a monograph treatment of the many extensions and applications of the ballot theorem, see Takács (1967). [Of course, there are implicit exchangeability conditions here: in an actual election - say, for parliament - the lead may fluctuate during the count because of the psephological characteristics of the particular constituency.]

A form of the ballot theorem arises in the context of skip-free random walks. Call a walk on the integer lattice $\mathbb{Z}$ *left-continuous*, or skip-free to the left, if the step-length law F is supported on $\{-1, 0, 1, 2, \cdots\}$ - thus the walk can jump to the right, but moves to the left continuously (which on the lattice means one step at a time). Right-continuous random walks are defined analogously. If T is the first- passage time from 0 to $-k$ in a left-continuous walk ($k = 1, 2, \cdots$), one has *Kemperman's identity*

$$P(T_k = n) = (k/n) P(S_n = -k) \qquad (1 \leq k \leq n)$$

(Kemperman (1961); for a simple proof, see Wendel (1975)).

Kemperman's identity allows one to prove quite simply that for skip-free random walks, Spitzer's condition is equivalent to a domain-or-attraction condition. Of course, this result is to be expected: we noted above that this equivalence holds 'far from symmetry', and skip-free random walks are 'completely asymmetrical'.

§**5. Queues.** The fluctuation theory of random walks developed above is immediately applicable to queues. We consider first the $GI/G/1$ queue (Kendall's notation: GI for general input, G for general service-time, 1 for the single server, Kendall (1951)); we follow Feller (1971), VI.9, Grimmett & Stirzaker (1992), Ch. 11, or Asmussen (1987), III.7, VII, IX.

Suppose customers (labelled $0, 1, 2, \cdots$) arrive for service at a single-server (free at time 0) at times $0, A_1, A_1 + A_2, \cdots$ (so the A_n are the *inter-arrival times*). Let B_{n+1} be the *service-time* of the nth customer. We assume the A_n are i.i.d. with law $A(.)$, the B_n i.i.d. with law $B(.)$, and the A_n and B_n are mutually independent. Write

$$X_n := B_n - A_n \qquad (n = 1, 2, \cdots),$$

and consider the random-walk $S_n := \sum_{k=1}^{n} S_k$ generated by the X_n. Let W_n be the *waiting-time* of the nth customer, for the queue to clear and his service to begin (so $W_0 = 0$, and $W_n = 0$ iff the nth customer is *lucky* - arrives to find the server free). Then considering the situations facing the nth and $(n + 1)$th customers on arrival, we see that

$$W_{n+1} = (W_n + B_{n+1} - A_{n+1})^+ = (W_n + X_{n+1})^+,$$

the *Lindley relation* (Lindley (1952).

Write a, b for the means of A, B, both assumed finite, and write $\rho := b/a$ for the *traffic intensity*. If $\rho < 1$, the mean service demand of a new customer is less than the mean time to the next arrival: it is then plausible, and true, that the queue is *stable* - settles down to an equilibrium state as time $t \to \infty$ (or as $n \to \infty$), irrespective of the initial conditions. If W has the limiting waiting-time distribution, the Lindley relation above suggests the equality in distribution

$$W = (W + X)^+,$$

where X has the law of the X_n. Writing $W(.)$, $F(.)$ for the distribution functions of W, X, this says

$$W(x) = \int_0^\infty F(x - y) dW(y) \qquad (x \geq 0),$$

an integral equation of Wiener-Hopf type. Its solution was analysed in detail by Spitzer (1957), who showed that there is a unique solution $W(.)$ which is a proper probability distribution ($W(\infty) = 1$) iff the traffic intensity $\rho < 1$ - that is, when the queue is stable. When $\rho \geq 1$, there is no such solution, waiting times tend to $+\infty$ in probability ($\rho > 1$), or are unbounded in probability ($\rho = 1$), and the queue is unstable.

The link between random walks and waiting-times is even stronger: one has equality of distribution between W_n, the waiting-time of the nth customer, and

$$M_n := \max\{0, S_1, \cdots, S_n\}$$

(this holds for each n separately, not jointly: observe that the sequence (M_n) is increasing, while (W_n) is not). To prove this (as in Feller (1971), VI.9), we note the following.
(i) The (strong) descending ladder indices of the random walk (S_n) correspond to the lucky customers who arrive to find the server free.
(ii) If $[n]$ denotes the index of the *last* ladder epoch up to time n,

$$W_n = S_n - S_{[n]}.$$

(iii) If $X_1, \cdots, X_n$ are written in reverse order as $X'_1, \cdots, X'_n$, with partial sums $S'_k = S_n - S_{n-k}$, and $M'_n = \max\{0, S'_1, \cdots, S'_n\}$, then

$$S_n - S_{[n]} = M'_n.$$

The result then follows as by symmetry M_n and M'_n have the same distribution.

As $n \to \infty$, $M_n \uparrow M < \infty$ iff the random walk (S_n) drifts to $-\infty$, that is, $EX_n = b - a < 0$, i.e. $b < a$, $\rho := b/a < 1$. We can read off the limiting distribution M of the M_n from Spitzer's identity, which contains the distributions of the M_n, to get

$$E \exp\{itM\} = \exp\{\sum_{n=1}^{\infty} \frac{1}{n}(E \exp\{itS_n^+\} - 1)\} \qquad (\rho < 1).$$

When means exist, the sign of $b - a$ discriminates between drift to $-\infty$, drift to $+\infty$ and oscillation; this tells us again the the limiting waiting-time law W exists if $\rho < 1$ but not otherwise.

For a stable queue, we can consider the number N of customers served in the first busy period (initiated by the arrival of the 0th customer at time 0) - or, by the same token, any busy period. Note that the busy period is a.s. finite - and so $N < \infty$ a.s. - iff the queue is stable. We can also consider the length T of the first (or any) busy period, and the length I of the first (or any) idle period. It turns out that the ladder variables of III.2 provide the key to this analysis, in view of the following.
(i) The number N of customers served in the first busy period is the first weak descending ladder epoch of (S_n).
(ii) The length I of the first idle period is given by $I = -S_N$, where S_N is the first weak descending ladder height of (S_n).
Thus (N, I) may be handled by the ladder methods discussed earlier. In fact, (N, T) may also be handled similarly: see Kingman (1962b), (1966).

Many-server queues. For the queue $GI/G/s$ with s servers, the theory is much more fragmentary when $s > 1$. Kiefer & Wolfowitz (1955), (1956) show that the appropriate

definition of the traffic intensity is now $\rho := b/(as)$: when $\rho < 1$, the queue is stable - the s-vector of the servers' virtual waiting times converges to equilibrium. They also obtain an analogue of Lindley's equation, and several other results. For background on many-server queues, see e.g. de Smit (1973b), (1973c) and the references cited there. A study of higher-dimensional random walks motivated by queueing theory has been given by Cohen (1992).

Lindley equations in more general contexts. Lindley equations have been studied in higher dimensions, and in contexts such as branching random walk (II, §7.4). For details, see Karpelevich, Kelbert and Suhov (1994), Biggins (1998).

§**6. Continuous time.** We saw above that the fluctuation theory of random walks (S_n) involved the maximum process (M_n), and also $(M_n - S_n)$, whose distributions are given in terms of the Wiener-Hopf factors of the random walk. In continuous time, the natural analogue of a random walk is a Lévy process $X = (X_t)$, whose distribution is specified by its Lévy exponent, $\psi(.)$ say:

$$E \exp\{sX_t\} = \exp\{t\psi(s)\} \qquad (\Re s = 0)$$

(here ψ is given by the Lévy-Khintchine formula in terms of its Lévy measure ν, which governs the jumps of the process X: see e.g. Bertoin (1996)). The analogues of (M_n), $(M_n - S_n)$ are $\bar{X}$, $\bar{X} - X$, where $\bar{X}$ is the supremum process:

$$\bar{X}_t := \sup\{X_s : 0 \le s \le t\}.$$

It turns out that both Spitzer's identity and the Spitzer-Baxter identity - or first and second factorization identities - have analogues for Lévy processes. For a Lévy process X with Lévy exponent ψ, and $\sigma > 0$, there is a *Wiener-Hopf factorization*

$$\sigma/(\sigma - \psi(s)) = \psi_\sigma^+(s)\psi_\sigma^-(s),$$

where
(i) $\psi_\sigma^+(s)$ is analytic in $\Re s < 0$, continuous and non-vanishing in $\Re s \le 0$, is the Laplace transform of an infinitely divisible probability law on the right half-line, and gives the distributions of $\bar{X}$:

$$\begin{aligned}\psi_\sigma^+(s) &= \sigma \int_0^\infty e^{-\sigma t} E \exp\{s\bar{X}_t\} dt \\ &= \exp\{\int_0^\infty (e^{sx} - 1) \int_0^\infty t^{-1} e^{-\sigma t} P(X_t \in dx) dt\},\end{aligned}$$

(ii) similarly for $\psi_\sigma^-(s)$ in $\Re s > 0$, $\Re s \geq 0$ and the left half-line, and $\psi_\sigma^-(s)$ gives the distributions of $\bar{X} - X$:

$$\begin{aligned}\psi_\sigma^-(s) &= \sigma \int_0^\infty e^{-\sigma t} E \exp\{s(X_t - \bar{X}_t)\} dt \\ &= \exp\{\int_{-\infty}^0 (e^{sx} - 1) \int_0^\infty t^{-1} e^{-\sigma t} P(X_t \in dx) dt\}.\end{aligned}$$

For proofs, see Greenwood (1975), (1976), Greenwood & Pitman (1980a), (1980b), and for the applied background, Bingham (1975).

The Wiener-Hopf factors $\psi_\sigma^\pm$ are thus the key to the fluctuation theory of Lévy processes X; however, given the Lévy exponent ψ of X it is not in general possible to evaluate the integrals above explicitly. But it is possible to do this in the *one-sided case*: when the Lévy measure - or *spectral measure* - ν is concentrated on one half-line. Such an X is called *spectrally positive* if ν vanishes on $(-\infty, 0)$ (X has no negative jumps), *spectrally negative* if ν vanishes on $(0, \infty)$ (X has no positive jumps) - see e.g. Bertoin (1996), VII for background. This is indeed fortunate: not only are the Wiener-Hopf factors available in this case (they may be evaluated in terms of the inverse function η of ψ: see e.g. Bingham (1975), Th. 4a), but it is just this case which is important in practice, as it occurs in the applied probability models of queues and dams, to which we turn below.

Splitting times. The last 'ladder epoch' -

$$\rho_t := \sup\{s \leq t : \bar{X}_s = X_s\},$$

the last time the supremum to date was attained - is an example of what is called a *splitting time*. It is far from a stopping time (ρ_t depends on all of $\sigma\{X_s : 0 \leq s \leq t\}$), and one is accustomed to use conditional independence at a *stopping* time, by the strong Markov property. Nevertheless, one can use ρ_t to *split* the path $\{X_s : 0 \leq s \leq t\}$ to date into the pre-ρ_t and post-ρ_t fragments, *and these are independent*. The use of splitting times in this context - fluctuation theory of Lévy processes - is due to Greenwood & Pitman (1980a). Splitting times were introduced by Williams (1974) (Brownian motion), Jacobsen (1974), Millar (1977a), (1977b) (Markov processes); for a textbook treatment, see Rogers & Williams (1994), III.49.

Queues and dams. The discrete-time framework of III.5 focusses on the individual customers. Suppose, however, that we focus on the server, and study his work-load (amount of service-demand in the system), as a function of time - which is continuous. It is now

more natural to use a stochastic process formulation throughout. For the $GI/G/1$ queue above, the input process (the point process of customer arrivals) is a *renewal process*; for the most important special case - when the inter-arrival time distribution A is exponential - this renewal process is a *Poisson process*, which is *Markovian* (and is the only renewal process with the Markov property). The queue is now called $M/G/1$, to emphasise the Markovian nature of the input stream. The cumulative service demand to date, $U(t)$, is a *compound Poisson process*; if $X_t := t - U_t$, $X = (X_t)$ is a spectrally negative Lévy process. If V_t denotes the *virtual waiting-time* at time t - the time that a hypothetical customer arriving at time t would have to wait, or the *service-load* facing the server - then

$$V_t = \bar{X}_t - X_t;$$

see Takács (1962) for this result, and for background. Thus, for instance, the server is idle when X has a ladder epoch. One can think of the pent-up service demand as being 'stored' in the queue, and this suggests that the queueing model above extends to other storage models, such as those of dams. This is indeed the case; see Bingham (1975) for details.

§**7. Barrier problems.** Suppose we are interested in the time and place of first passage to or over a positive barrier x, starting at 0. The first-passage time process $\tau = (\tau_x)$ may be analysed together with the maximum process $M = (M_n)$, as these are pathwise inverse: $M_n \geq x$ iff $\tau_x \leq n$. The Wiener-Hopf factors needed above to handle M_n suffice also to handle τ_x. One is dealing here with random walks on a half-line (Spitzer (1964), IV).

Suppose instead one starts with a random walk at the origin, and runs it till it first exits from an *interval* $[-y, x]$ $(0 < x, y)$ containing the origin. Such two-barrier problems are harder: one has here a random walk on an interval (Spitzer (1964), V).

A detailed treatment of such first-passage problems for Markov processes is given by Kemperman (1961). Results of this type are relevant to *sequential analysis* in statistics, where one samples until the test statistic exits from an interval, accepting one of two hypotheses depending on which barrier exit is across. For background, see e.g. Shiryaev (1973), Ch. IV.

Similar one- and two-barrier problems arise in continuous time for Lévy processes (for a Wiener-Hopf formulation of the one-barrier case, see e.g. Bingham (1975), Th. 1e). They have applications to queues and dams: queues with finite waiting-room, and finite dams - which may overflow as well as be empty. For such applied background, see e.g.

Bingham (1975), §7 and the references there, particularly to the works of Takács. For a very simple proof of the principal explicit result in this area, see Rogers (1990).

§**8. Higher dimensions and algebraic extensions.** The classical results of Part I on random walk in one dimension generalise, for the most part, to higher dimensions. It is natural to ask whether this is true of the fluctuation theory of Part III. The situation here is clearly different and less positive: we deal here with the *maximum* partial sum, and the maximum is with respect to the total ordering on the real line $\mathbb{R}$. For $\mathbb{R}^d$, $d \geq 2$, no such total ordering exists - though partial orderings do.

The half-line $[0, \infty)$ plays a key role in Spitzer's identity on $\mathbb{R}$ - via the positive parts S_n^+ of the partial sums S_n - and the key property of the half-line relevant here is that of being a *convex cone* (closed under vector addition and multiplication by non-negative scalars). A close analogue of the Spitzer-Baxter identity, giving the joint distribution of the time and place of first exit of a random walk in $\mathbb{R}^d$ from a convex cone, was given by Greenwood & Shaked (1977), who gave applications to queueing and storage systems in d dimensions (see also Mogulskii & Pecherskii (1977)). This joint law is given in terms of what one calls a *Wiener-Hopf factor* for the cone, by analogy with the one-dimensional case. Now in d dimensions for $d \geq 2$, the number of convex cones needed to fill out the whole of $\mathbb{R}^d$ is greater than two, *except* for the case of two cones which are complementary half-spaces. This situation is really one-dimensional, on projecting onto the normal through the origin to the hyperplane bounding the complementary half-spaces. Thus a genuinely higher-dimensional fluctuation theory requires at least three Wiener-Hopf factors (Kingman (1962a) observed that a two-factor theory must be essentially one-dimensional, as above; the multi-factor theory was later developed by Greenwood & Shaked (1977)).

The Sparre Andersen equivalence principle, however, does not extend from one to higher dimensions. For details, see e.g. Hobby & Pyke (1963a), (1963b), (1963c), Pyke (1973), §4.3.2.

The algebra of queues. One of the shortest proofs of Spitzer's identity is that of Wendel (1958). Kingman (1966), §§2, 3 gave a systematic treatment of the algebraic structure of this and related results, isolating the concept of a *Wendel projection*. Kingman (1966) also discusses the Wiener-Hopf technique in this connection (§6), combinatorial aspects (§7: these go back to Spitzer (1956)), and the concept of a *Baxter algebra* (§13). Kingman's algebraic approach is primarily motivated by the theory of the single-server queue above;

the case of a many-server queue, which as we saw in III.5 is much harder, is discussed briefly in Kingman (1966), §12.

Combinatorics on words. The algebraic and combinatorial aspects of Spitzer's identity and related results have also been studied in connection with the subject of combinatorics on words. The connection is due to Foata & Schützenberger (1971), and has been developed in the book Lothaire (1983) (Lothaire is the pen-name of a group of mathematicians including Schützenberger and co-workers). See in particular Chapter 5 there by Perrin (the combinatorial content of Spitzer's identity is Th. 5.4.3, and Sparre Andersen's equivalence principle is Prop. 5.2.9), and Chapter 10 by Foata.

§9. Distribution-free results and non-parametric statistics.

Empiricals. Recall the classical setting of the Kolmogorov-Smirnov test of non-parametric statistics. We draw a random sample of size n from a population distribution F; we use the n readings X_i to form the *empirical distribution function*

$$F_n(x) := \frac{1}{n}\sum_{k=1}^{n} I(X_k \leq n);$$

thus F_n has a jump of size $1/n$ at each *order statistic* - the readings arranged in increasing order. By the Glivenko-Cantelli theorem (or Fundamental Theorem of Statistics), F_n converges to F uniformly on the whole line, with probability one. Thus if

$$D_n := \sup\{|F_n(x) - F(x)| : x \in \mathbb{R}\}$$

is the *discrepancy* between F_n and F,

$$D_n \to 0 \qquad (n \to \infty) \qquad \text{a.s.}$$

(the one-sided version $D_n^+ := \sup\{F_n(x) - F(x) : x \in \mathbb{R}\}$ is also useful). To test the hypothesis that the (unknown) population distribution is some specified F, we need to know the distribution of D_n under this hypothesis. Provided only that F is *continuous* (so the readings are all distinct, and the order statistics defined unambiguously), the distribution of D_n is the *same* for all F, and so has a *distribution-free* character (like that of the discrete arc-sine law of III.3). This enables one to use the above to construct a non-parametric test for the null hypothesis that the population distribution is F. For the distribution-free nature of D_n, see e.g. Feller (1968), III.1 Example (c), Feller (1971), I.12. For the limit distribution of $\sqrt{n}D_n$, see Feller (1971), X.6, or Billingsley (1968) (Billingsley uses weak convergence theory to derive this limit distribution in terms of the

Brownian bridge). This test - the Kolmogorov-Smirnov test - is one of the corner-stones of non-parametric statistics; for background, see e.g. Shorack & Wellner (1986).

One can define statistics D_n, D_n^+ of Kolmogorov-Smirnov type in $\mathbb{R}^d$, but for $d \geq 2$ the distribution-free character is lost (Simpson (1951)). Nevertheless, the limit distributions of D_n, D_n^+ are known (Kiefer & Wolfowitz (1958)).

Greatest convex minorants. If $X_1, \cdots, X_n$ are i.i.d., Sparre Andersen (1953/4), II used his results on fluctuation theory to show that the number of sides of the greatest convex minorant (GCM) of the graph $\{k, \sum_{j=1}^k X_j\}_1^n$ is the same as that of the number of cycles in a randomly chosen permutation on n objects - and thus, the GCM statistic is *distribution-free*. This distribution is also that of the number of (upper) *records* in $(X_1, \cdots, X_n)$ (Foster & Stuart (1954)), and is given by

$$p_r = |S_n^r|/n! \qquad (r = 1, \cdots, n),$$

where $|S_n^r|$ is the modulus of the Stirling number of the first kind (the coefficient of z^r in $z(z+1).\cdots.(z+r-1)$). A survey of combinatorial results of such kinds and their statistical applications is given by Barton & Mallows (1965).

In an improving population, records become more frequent, and so the GCM statistic is informative for *tests for trend*: testing a null hypothesis H_0 against an alternative hypothesis H_1, where

$$H_0: \qquad \mu_1 = \cdots = \mu_n, \qquad H_1: \qquad \mu_1 \leq \cdots \leq \mu_n;$$

here μ_i is the mean of X_i (Brunk (1960), using Sparre Andersen's result). There is a similar test based on medians (Brunk (1964), using Spitzer's combinatorial lemma). For textbook accounts of such statistical inference under *order restrictions*, see Barlow *et al.* (1972), Robertson *et al.* (1988). Problems of this type are topical in environmental statistics and studies of global warming, for example.

§**10. Postscript.** As we have seen, fluctuation theory in one dimension is remarkably well-developed and complete, as regards both theory and applications. By contrast, the situation in higher dimensions is much less complete, and our knowledge here remains fragmentary by comparison. Study of the higher-dimensional case has several motivations, of which we mention three here to close: mathematical interest, and the needs of non-parametric statistics in higher dimensions and the theory of queues with many servers.

REFERENCES

Aldous, D. (1983): Random walk on finite groups and rapidly mixing Markov chains. *Séminaire de Probabilités XVII*, 243-297. *Lecture Notes in Math.* **986**, Springer.

Aldous, D. (1989): An introduction to covering problems for random walks on graphs. *J. Theor. Probab.* **2**, 87-89.

Aldous, D. & Diaconis, P. (1986): Shuffling cards and stopping times. *American Math. Monthly* **93**, 333-348.

Aldous,D. & Fill, J. A. (1998+): *Reversible Markov chains and random walks on graphs*, book to appear.

Ancona, A. (1990): Théorie du potentiel sur les graphes et les variétés. *École d'Été de Probabilités de Saint-Flour XVIII*, 3-112, *Lecture Notes in Math.* **1427**, Springer.

Andersen, E. Sparre (1953/4): On the fluctuations of sums of random variables, I, II. *Math. Scand.* **1**, 263-285, **2**, 195-223.

Arjas, E. & Speed, T. P. (1973): Symmetric Wiener-Hopf factorizations in Markov additive processes. *Z. Wahrschein.* **26**, 105-118.

Asmussen, S. (1982): Conditional limit theorems relating a random walk to its associate, with applications to risk reserve processes and the $GI/G/1$ queue. *Adv. Appl. Probab.* **14**, 143-170.

Asmussen, S. (1987): *Applied probability and queues.* Wiley.

Asmussen, S. (1989): Aspects of matrix Wiener-Hopf factorization in applied probability. *Math. Scientist* **14**, 101-116.

Baldi, P. (1981): Charactérisation des groupes de Lie connexes récurrents. *Ann. Inst. H. Poincaré, Sér. Prob. Stat.* **17**, 281-308.

Ball, F. G., Dunham, B. & Hirschowitz, A. (1997): On the mean and variance of cover times for random walks on graphs. *J. Math. Anal. Appl.* **207**, 506-514.

Barber, M. N. & Ninham, B. W. (1970): *Random and restricted random walks: Theory and applications.* Gordon & Breach, New York.

Barlow, R. E., Bartholomew, D. J., Bremner, J. M. & Brunk, H. D. (1972): *Statistical inference under order restrictions.* Wiley.

Barton, D. E. & Mallows, C. L. (1965): Some aspects of the random sequence. *Ann. Math. Statist.* **36**, 236-260.

Baxter, G. (1958): An operator identity. *Pacific J. Math.* **4**, 649-663.

Bayer, D. & Diaconis, P. (1992): Trailing the dovetail shuffle to its lair. *Ann. Appl. Probab.* **2**, 294-313.

Beardon, A. F. (1983): *The geometry of discrete groups.* Graduate Texts in Math. **91**,

Springer.
Belsley, E. D. (1998): Rates of convergence of random walk on distance-regular graphs. *Probab. Theor. Rel. Fields* **112**, 493-533.
Bertoin, J. (1996): *Lévy processes*. Cambridge Tracts in Math. **121**, Cambridge University Press.
Bertoin, J. & Doney, R. A. (1994): On conditioning a random walk to stay non-negative. *Annals of Probability* **22**, 2152-2167.
Biggins, J. D. (1995): The growth and spread of the general branching random walk. *Ann. Appl. Probab.* **5**, 1008-1024.
Biggins, J. D. (1998): Lindley-type equations in the branching random walk. *Stochastic Processes and Applications* **75**, 105-133.
Biggins, J. D. & Bingham, N. H. (1991): Near-constancy phenomena for supercritical branching processes. *Math. Proc. Cambridge Phil. Soc.* **110**, 545-558.
Biggins, J. D. & Bingham, N. H. (1993): Large deviations in the supercritical branching process. *Adv. Appl. Probab.* **25**, 757-772.
Biggs, N. L. (1974): *Algebraic graph theory*. Cambridge Tracts in Math. **67**, Cambridge Univ. Press (2nd ed. 1993).
Biggs, N. L. (1997): Algebraic potential theory on graphs. *Bull. London Math. Soc.* **29**, 641-682.
Billingsley, P. (1968): *Convergence of probability measures*. Wiley.
Bingham, N. H. (1972): Random walk on spheres. *Z. Wahrschein. verw. Geb.* **22**, 169-192.
Bingham, N. H. (1973): Limit theorems for a class of Markov processes. *Stochastic Analysis* (Rollo Davidson Memorial Volume, ed. Kendall, D. G. & Harding, E. F.) 266-293, Wiley.
Bingham, N. H. (1975): Fluctuation theory in continuous time. *Adv. Appl. Probab.* **7**, 705-766.
Bingham, N. H. (1980): Wiener-Hopf and related methods in probability theory. *Aspects of Contemporary Complex Analysis* (ed. Brannan, D. A. & Clunie, J. G.) 369-375, Academic Press.
Bingham, N. H. (1989): Tauberian theorems in probability theory. *Probability on Groups IX*, 6-20. *Lecture Notes in Math.* **1379**, Springer.
Bingham, N. H. (1991): Fluctuation theory for the Ehrenfest urn. *Adv. Appl. Probab.* **23**, 598-611.
Bingham, N. H. (1998): Fluctuations. *Math. Scientist* **23**, 63-73.
Bingham, N. H., Goldie, C. M. & Teugels, J. L. (1987): *Regular variation*. Encycl. Math.

Appl. **27**, Cambridge Univ. Press.
Bingham, N. H. & Kiesel, R. (1998): *Risk-neutral valuation: Pricing and hedging of financial derivatives*, Springer, London.
Blackwell, D. (1953): Extension of a renewal theorem. *Pacific J. Math.* **3**, 315-320.
Bloom, W. & Heyer, H. (1994): *Harmonic analysis of probability measures on hypergroups*. De Gruyter Studies in Math. **20**, Walter de Gruyter.
Bondar, J. and Milnes, P. (1981): Amenability: a survey for statistical applications of Hunt-Stein and related conditions on groups. *Z. Wahrschein.* **57**, 103-128.
Broder, A. & Karlin, A. R. (1989): Bounds on the cover time. *J. Theor. Probab.* **2**, 101-120.
Brunk, H. D. (1960): A theorem of E. Sparre Andersen and its application to tests against trend. *Math. Scand.* **8**, 305-326.
Brunk, H. D. (1964): A generalization of Spitzer's combinatorial lemma. *Z. Wahrschein. verw. Geb.* **2**, 395-405.
Chavel, I. (1993): *Riemannian geometry: A modern introduction*. Cambridge Tracts in Math. **108**, Cambridge Univ. Press.
Chung, K.-L. (1962): On the renewal theorem in higher dimensions. *Skand. Akt.* **35**, 188-194.
Chung, K.-L. (1967): *Markov chains with stationary transition probabilities*. Grundlehren math. Wiss. **104**, Springer.
Chung, K.-L. (1968): *A course in probability theory*. Harcourt, Brace & Jovanovich Inc. (2nd ed. 1974, Academic Press).
Chung, K.-L. (1995): *Green, Brown and probability*. World Scientific.
Chung, K.-L. & Feller, W. (1949): On fluctuations in coin-tossing. *Proc. Nat. Acad. Sci. USA* **35**, 605-608.
Chung, K.-L. & Fuchs, W. H. J. (1951): On the distribution of values of sums of random variables. *Memoirs Amer. Math. Soc.* **6**, 12p.
Cohen, J. W. (1992): *Analysis of random walks*. Studies in Probability, Optimisation and Statistics **2**, IOS Press, Amsterdam.
Cox, J. C., Ross, S. A. & Rubinstein, M. (1979): Option pricing: a simplified approach. *J. Financial Economics* **7**, 229-263.
Coxeter, H. S. M. (1963): *Regular polytopes*, Macmillan (3rd ed. 1973, Dover).
Davis, B. (1989): Loss of recurrence in reinforced random walk. *Almost Everywhere Convergence* (ed. Edgar, G. A. & Sucheston, L.) 179-188, Academic Press.
Davies, E. B. (1989): *Heat kernels and spectral theory*. Cambridge Tracts in Math. **92**, Cambridge Univ. Press.

Davies, E. B. (1993): Large deviations for heat kernels on graphs. *J. London Math. Soc.* (2) **47**, 65-72.

Dembo, A. & Zeitouni, O. (1993): *Large deviations techniques and applications*. Jones & Bartlett, London & Boston.

Derriennic, Y. & Guivarc'h, Y. (1973): Théorème de renouvellement pour les groupes non moyennables. *Comptes Rendues Hebd. Acad. Sci.* **277A**, 613-615.

Deuschel, J.-D. & Stroock, D. (1989): *Large deviations*. Academic Press.

Devroye, L. & Sbihi, A. (1990): Random walks on highly symmetric graphs. *J. Theor. Probab.* **4**, 497-514.

Diaconis, P. (1988): *Group representations in probability and statistics*. IMS Lecture Notes **11**, Inst. Math. Statist., Hayward, CA.

Diaconis, P., McGrath, M. & Pitman, J. W. (1995): Riffle shuffles, cycles and descents. *Combinatorica* **15**, 11-29.

Diaconis, P. & Stroock, D. (1991): Geometric bounds for eigenvalues of Markov chains. *Ann. Appl. Probab.* **1**, 36-61.

Dodziuk, J. (1984): Difference equations, isoperimetric inequality and transience of certain random walks. *Trans. Amer. Math. Soc.* **284**, 787-794.

Dodziuk, J. & Kendall, W. S. (1986): Combinatorial Laplacians amd isoperimetric inequality. *From local times to global geometry, control and physics* (ed. Elworthy, K. D.) 68-74. Research Notes in Math. **150**, Pitman.

Doeblin, W. (1938): Exposé de la théorie des chaînes simples constantes de Markov à un nombre fini d'états. *Rev. Math. Union Interbalkanique* **2**, 77-105.

Doney, R. A. (1995): Spitzer's condition and ladder variables in random walks. *Probab. Th. Rel. Fields* **101**, 577-580.

Doyle, P. G. & Snell, J. L. (1984): *Random walks and electric networks*. Carus Math. Monographs **22**, Math. Assoc. America, Washington DC.

Dwass, M. & Karlin, S. (1963): Conditioned limit theorems. *Ann. Math. Statist.* **34**, 1147-1167.

Dynkin, E. B. (1961): Limit theorems for sums of independent random variables with infinite expectation. *Selected Translations (IMS-AMS)* **1**, 171-189.

Elie, L. (1982a): Comportement asymptotique du noyau potentiel sur les groups de Lie. *Ann. Sci. Éc. Norm. Sup.* (4) **15**, 257-364.

Elie, L. (1982b): Sur le théorème de dichotomie pour une marche aléatoire sur un espace homogène. *Probability on Groups VI*, 60-75. *Lecture Notes in Math.* **928**, Springer.

Eymard, P. (1972): *Moyennes invariantes et représentations unitaires*. Lecture Notes in Math. **300**, Springer.

Falconer, K. J. (1985): *The geometry of fractal sets.* Cambridge Tracts in Math. **85**, Cambridge Univ. Press.

Feller, W. (1949): Fluctuation theory of recurrent events. *Trans. Amer. Math. Soc.* **67**, 98-119.

Feller, W. (1968): *An introduction to probability theory and its applications, Volume 1*, 3rd ed., Wiley (1st ed. 1950, 2nd ed. 1957).

Feller, W. (1971): *An introduction to probability theory and its applications, Volume 2*, 2nd ed. (1st ed. 1966).

Fill, J. A. (1991): Eigenvalue bounds on convergence to stationarity for nonreversible Markov chains, with an application to the exclusion process. *Ann. Appl. Probab.* **1**, 62-87.

Foata, D. & Schützenberger, M.-P. (1971): On the principle of equivalence of Sparre Andersen. *Math. Scand.* **28**, 308-316.

Foster, F. G. & Stuart, A. (1954): Distribution-free tests in time series based on the breaking of records. *J. Roy. Statist. Soc. B* **16**, 1-22.

Furstenberg, H. (1971): Random walks and discrete subgroups of Lie groups. *Advances in Probability and Related Topics* **1** (ed. Ney, P. E.), 1-63.

Gerl, P. (1988): Random walks on graphs with a strong isoperimetric inequality. *J. Theoretical Probability* **1**, 171-187.

Gerl, P. & Woess, W. (1986): Simple random walks on trees. *European J. Combinatorics* **7**, 321-331.

Guivarc'h, Y., Keane, M. & Roynette, B. (1977): *Marches aléatoires sur les groupes de Lie.* Lecture Notes in Math. **624**, Springer.

Grabner, P. J. (1997): Functional iteration and stopping times for Brownian motion on the Sierpinski gasket. *Mathematika* **44**, 374-400.

Grabner, P. J. & Woess, W. (1997): Functional iterations and periodic oscillations for simple random walk on the Sierpinski graph. *Stoch. Proc. Appl.* **69**, 127-138.

Greenleaf, F. P. (1969): *Invariant means on topological groups and their applications.* Van Nostrand Math. Studies **19**, Van Nostrand, New York.

Greenwood, P. E. (1975): Wiener-Hopf methods, decompositions and factorisation identities for maxima and minima of homogeneous random processes. *Adv. Appl. Probab.* **7**, 767-785.

Greenwood, P. E. (1976): Wiener-Hopf decomposition of random walks and Lévy processes. *Z. Wahrschein verw. Geb.* **34**, 193-198.

Greenwood, P. E. & Pitman, J. W. (1980a): Fluctuation theory for Lévy processes and splitting at the maximum. *Adv. Appl. Probab.* **12**, 893-902.

Greenwood, P. E. & Pitman, J. W. (1980b): Construction of local time and Poisson point processes from nested arrays. *J. London Math. Soc.* (2) **22**, 182-192.
Greenwood, P. E. & Shaked, M. (1977): Fluctuations of random walk in $\mathbb{R}^d$ and storage systems. *Adv. Appl. Probab.* **9**, 566-587.
Greenwood, P. E. & Shaked, M. (1978): Dual pairs of stopping times for random walks. *Ann. Probab.* **6**, 644-650.
Grimmett, G. R. & Stirzaker, D. R. (1992): *Probability and random processes*, 2nd ed., Oxford Univ. Press (1st ed. 1982).
Hattori, K. & Hattori, T. (1991): Self-avoiding process on the Sierpinski gasket. *Prob. Th. Rel. Fields* **88**, 405-428.
Hattori, K, Hattori, T. & Kusuoka, S. (1990): Self-avoiding paths on the pre-Sierpinski gasket. *Prob. Th. Rel. Fields* **84**, 1-26.
Helgason, S. (1962): *Differential geometry and symmetric spaces.* Academic Press.
Heyer, H. (1977): *Probability measures on locally compact groups.* Ergebnisse Math. **94**, Springer.
Heyer, H. (1983): Convolution semigroups of probability measures on Gelfand pairs. *Expositiones Math.* **1**, 3-45.
Heyde, C. C. (1969): On extremal factorisation and recurrent events. *J. Roy. Statist. Soc. B* **31**, 72-79.
Hobby, C. R. & Pyke, R. (1963a): Combinatorial results in multi-dimensional fluctuation theory. *Ann. Math. Statist.* **34**, 402-404.
Hobby, C. R. & Pyke, R. (1963b): Combinatorial results in fluctuation theory. *Ann. Math. Statist.* **34**, 1233-1242.
Hobby, C. R. & Pyke, R. (1963c): Remarks on the equivalence principle in fluctuation theory. *Math. Scand.* **12**, 19-24.
Hughes, B. D. (1995): *Random walks and random environments. Volume 1: Random walks.* Oxford Univ. Press.
Jacobsen, M. (1974): Splitting times for Markov processes and a generalised Markov property for diffusions. *Z. Wahrschein. verw. Geb.* **30**, 27-43.
Jones, O. D. (1996): Transition probabilities for the simple random walk on the Sierpinski gasket. *Stoch. Proc. Appl.* **61**, 45-69.
Kac, M. (1959): *Probability and related topics in the physical sciences.* Interscience.
Kahn, J. D., Linial, N., Nisan, N. & Saks, M. E. (1989): On the cover time of random walks on graphs. *J. Th. Probab.* **2**, 121-128.
Kakutani, S. (1944): Two-dimensional Brownian motion and harmonic functions. *Proc. Imp. Acad. Tokyo* **20**, 706-714.

Kaimanovich, V. A. (1991): Poisson boundaries of random walks on discrete solvable groups. *Probability on Groups X* (ed. H. Heyer), 39-43, Plenum Press.
Kaimanovich, V. A. & Vershik, A. (1983): Random walk on discrete groups: boundary and entropy. *Ann. Probab.* **11**, 457-490.
Karpelevich, F. I., Kelbert, M. Ya. & Suhov, Yu. M. (1994): Higher-order Lindley equations. *Stoch. Proc. Appl.* **53**, 65-96.
Kelly, F. P. (1979): *Reversibility and stochastic networks.* Wiley.
Kemperman, J. H. B. (1961): *The passage problem for a stationary Markov chain.* Univ. Chicago Press.
Kendall, D. G. (1951): Some problems in the theory of queues. *J. Roy. Statist. Soc. B* **13**, 151-185.
Kennedy, J. E. (1998): On duality and the Spitzer-Pollaczek factorization for random walks. *Stoch. Proc. Appl.* **76**, 251-266.
Kesten, H. (1959): Symmetric random walks on groups. *Trans. Amer. Math. Soc.* **92**, 336-354.
Kesten, H. (1967): The Martin boundary of recurrent random walks on countable groups. *Proc. Fifth Berkeley Symp. Math. Statist. Probab. Volume II*, 51-74, Univ. California Press.
Kesten, H. & Spitzer, F. L. (1965): Random walks on countably infinite Abelian groups. *Acta Math.* **114**, 237-265.
Kiefer, J. & Wolfowitz, J. (1955): On the theory of queues with many servers. *Trans. Amer. Math. Soc.* **78**, 1-18.
Kiefer, J. & Wolfowitz, J. (1956): On the characteristics of the general queueing process, with applications to random walks. *Ann. Math. Statist.* **27**, 147-161.
Kiefer, J. & Wolfowitz, J. (1958): On the deviations of the empirical distribution of vector chance variables. *Trans. Amer. Math. Soc.* **87**, 173-186.
Kingman, J. F. C. (1962a): Spitzer's identity and its use in probability theory. *J. London Math. Soc.* **37**, 309-316.
Kingman, J. F. C. (1962b): The use of Spitzer's identity in the investigation of the busy period and other quantities in the queue $GI/G/1$. *J. Australian Math. Soc.* **2**, 345-356.
Kingman, J. F. C. (1966): On the algebra of queues. *J. Applied Probability* **3**, 285-326 (reprinted as Volume **6** of Methuen's Review Series in Applied Probability).
Lamperti, J. (1962): An invariance principle in renewal theory. *Ann. Math. Statist.* **33**, 685-696.
Letac, G. (1981): Problèmes classiques de probabilité sur un couple de Gelfand. *Analytic Methods in Probability*, 93-120. *Lecture Notes in Math.* **861**, Springer.

Letac, G. (1982): Les fonctions sphériques d'un couple de Gelfand symmétrique et les chaînes de Markov. *Adv. Appl. Probab.* **14**, 272-294.

Letac, G. & Takács, L. (1980a): Random walk on a dodecahedron. *J. Appl. Probab.* **17**, 373-384.

Letac, G. & Takács, L. (1980b): Random walk on a 600-cell. *SIAM J. Alg. Discrete Methods* **1**, 114-120.

Lindley, D. V. (1952): The theory of queues with a single server. *Proc. Cambridge Math. Soc.* **48**, 277-289.

Lindvall, T. (1992): *Lectures on the coupling method.* Wiley.

Lothaire, M. (1983): *Combinatorics on words.* Encycl. Math. Appl. **17**, Addison-Wesley.

Lovász, L. (1986): Random walks on graphs: a survey. *Combinatorics. Paul Erdös is eighty. Volume 2*, 353-397. Bolyai Soc. Math. Studies **2**, Janos Bolyai Math. Soc., Budapest.

Lovász, L. & Winkler, P. M. (1995): Mixing of random walks and other diffusions on a graph. *Surveys in Combinatorics* (ed. Rowlinson, P.) 119-154, *London Math. Soc. Lecture Notes Series* **218**, Cambridge Univ. Press.

Lyons, R. (1990): Random walk and percolation on trees. *Ann. Probab.* **18**, 931-958.

Lyons, R. (1992): Random walks, capacity and percolation on trees. *Ann. Probab.* **20**, 2043-2088.

Lyons, R. & Pemantle, R. (1992): Random walk in a random environment and first-passage percolation on trees. *Ann. Probab.* **20**, 125-136.

Lyons, R. & Peres, Y. (1998+): *Probability on trees*, book in preparation.

Lyons, T. J. (1983): A simple criterion for transience of a reversible Markov chain. *Ann. Probab.* **11**, 393-402.

Lyons, T. J. & Sullivan, D. (1984): Function theory, random paths and covering spaces. *J. Differential Geometry* **19**, 299-323.

Madras, N. & Slade, G. (1993): *The self-avoiding walk.* Birkhäuser, Boston.

Millar, P. W. (1977a): Zero-one laws and the minimum of a Markov process. *Trans. Amer. Math. Soc.* **226**, 365-391.

Millar, P. W. (1977b): Random times and decomposition theorems. *Proc. Symp. Pure Math.* **31**, 91-103.

Mogulskii, A. A. & Pecherskii, E. A. (1977): On the first exit time from a semigroup in $\mathbb{R}^m$ for a random walk. *Th. Probab. Appl.* **22**, 818-825.

Motwani, R. & Raghavan, P. (1995): *Randomised algorithms.* Cambridge Univ. Press.

Nagaev, A. V. (1979): Renewal theorems in $\mathbb{R}^d$. *Th. Probab. Appl.* **24**, 572-581.

Nash-Williams, C. St. J. A. (1959): Random walks and electrical currents in networks.

Proc. Cambridge Phil. Soc. **55**, 181-194.
Norris, J. R. (1997): *Markov chains.* Cambridge Univ. Press.
Ornstein, D. (1969): Random walks, I, II. *Trans. Amer. Math. Soc.* **138**, 1-43, 45-60.
Palacios, J. L. (1993): Fluctuation theory for the Ehrenfest urn via electrical networks. *Adv. Appl. Probab.* **25**, 472-476.
Palacios, J. L. (1994): Another look at the Ehrenfest urn via electric networks. *Adv. Appl. Probab.* **26**, 820-824.
Palacios, J. L. & Renum, J. M. (1998): Random walks on edge-transitive graphs. *Statist. Probab. Letters* **37**, 29-34.
Paley, R. E. A. C. & Wiener, N. (1934): *Fourier transforms in the complex domain.* AMS Colloquium Publications **19**, Amer. Math. Soc., Providence, RI.
Pang, M. M. H. (1993): Heat kernels on graphs. *J. London Math. Soc.* (2) **47**, 50-64.
Pemantle, R. (1988): Phase transition in reinforced random walk. *Ann. Probab.* **16**, 1229-1241.
Pemantle, R. & Peres, Y. (1995): Critical random walk in random environment on trees. *Ann. Probab.* **23**, 105-140.
Picardello, M. A. & Woess, W. (1987): Martin boundaries of random walks: ends of trees and groups. *Trans. Amer. Math. Soc.* **302**, 185-205.
Pólya, G. (1921): Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Strassennetz. *Math. Annalen* **84**, 149- 160.
Port, S. C. (1963): An elementary approach to fluctuation theory. *J. Math. Anal. Appl.* **6**, 109-151.
Port, S. C. & Stone, C. J. (1969): Potential theory of random walks on abelian groups. *Acta Math.* **122**, 19-114.
Port, S. C. & Stone, C. J. (1978): *Brownian motion and classical potential theory.* Academic Press, New York.
Pyke, R. (1973): Partial sums of matrix arrays, and Brownian sheets. *Stochastic Analysis* (Rollo Davidson Memorial Volume, ed. Kendall, D. G. & Harding, E. F.) 349-363, Wiley.
Révész, P. (1990): *Random walks in random and non-random environments.* World Scientific.
Revuz, D. (1984): *Markov chains.* North-Holland, Amsterdam.
Rogers, L. C. G. (1985): Recurrence of additive functionals of Markov chains. *Sankhyā A* **47**, 47-56.
Rogers, L. C. G. (1990): The two-sided exit problem for spectrally positive Lévy processes. *Adv. Appl. Probab.* **22**, 486-487.
Rogers, L. C. G. & Williams, D. (1994): *Diffusions, Markov processes and martingales.*

Volume One: Foundations, 2nd ed., Wiley (1st ed., Williams, D., 1979).
Rudin, W. (1952): *Fourier analysis on groups*. Wiley/Interscience.
Sawyer, S. A. (1997): Martin boundaries and random walks. *Contemp. Math.* **206**, 17-44.
Sawyer, S. A. & Steger, T. (1987): The rate of escape for anisotropic random walks in a tree. *Prob. Th. Rel. fields* **76**, 207-230.
Schott, R. (1984): Random walks on homogeneous spaces. *Probability on Groups VII*, 564-575. *Lecture Notes in Math.* **1064**, Springer.
Schwarzenberger, R. (1974): The 17 plane symmetry groups. *Math. Gazette* **58**, 123-131.
Schwarzenberger, R. (1980): *N-dimensional crystallography*. Research Notes in Math. **41**, Pitman.
Series, C. (1983): Martin boundaries of random walks on Fuchsian groups. *Israel J. Math.* **44**, 221-242.
Shiryaev, A. N. (1973): *Sequential statistical analysis. Optimal stopping rules*. Transl. Math. Monographs **38**, Amer. Math. Soc., Providence, RI (2nd ed. *Optimal stopping rules*, 1978, Springer).
Shorack, G. R. & Wellner, J. A. (1986): *Empirical processes with applications to statistics*. Wiley, New York.
Simpson, P. B. (1951): Note on the estimation of a bivariate distribution function. *Ann. Math. Statist.* **22**, 476-478.
Sinai, Ya. G. (1957): On the distribution of the first positive sum for a sequence of independent random variables. *Th. Probab. Appl.* **2**, 122-129.
Sinclair, A. J. (1993): *Algorithms for random number generation and counting*.
Birkhäuser, Basel.
Smit, J. H. A. de (1973a): A simple analytic proof of the Pollaczek-Wendel identity for partial sums of order statistics. *Ann. Probab.* **1**, 348-351.
Smit, J. H. A. de (1973b): Some general results for many-server queues. *Adv. Appl. Probab.* **5**, 153-169.
Smit, J. H. A. de (1973c): On many-server queues with exponential service-times. *Adv. Appl. Probab.* **5**, 170-182.
Soardi, P. M. (1990): Recurrence and transience of the edge graph of a tiling in the Euclidean plane. *Math. Annalen* **287**, 613-626.
Spitzer, F. L. (1956): A combinatorial lemma and its application to probability theory. *Trans. Amer. Math. Soc.* **82**, 323-343.
Spitzer, F. L. (1957): The Wiener-Hopf equation whose kernel is a probability density. *Duke Math. J.* **24**, 327-343.

Spitzer, F. L. (1960): A Tauberian theorem and its probability interpretation. *Trans. Amer. Math. Soc.* **94**, 150-169.

Spitzer, F. L. (1964): *Principles of random walk.* Van Nostrand, Princeton, NJ (2nd ed., Graduate Texts in Math. **34**, 1976, Springer).

Sunyach, C. (1981): Capacité et théorie de renouvellement. *Bull. Soc. Math. France* **109**, 283-296.

Takács, L. (1962): *Introduction to the theory of queues.* Oxford Univ. Press, New York.

Takács, L. (1967): *Combinatorial methods in the theory of stochastic processes.* Wiley.

Takács, L. (1981): Random flights on regular polytopes. *SIAM J. Alg. Discrete Methods* **2**, 153-171.

Takács, L. (1982): Random walks on groups. *Linear Alg. Appl.* **43**, 49-67.

Takács, L. (1983): Random walk on a finite group. *Acta Sci. Math. (Szeged)* **45**, 395-408.

Takács, L. (1984): Random flights on regular graphs. *Adv. Appl. Probab.* **16**, 618-637.

Takács, L. (1986): Harmonic analysis on Schur algebras and its applications in the theory of probability. *Probability Theory in Harmonic Analysis* (ed. Chao, J.-A. & Woyczyński, W. A.) 227-283, Marcel Dekker, New York.

Takacs, C. (1998): Biased random walks on directed trees. *Probab. Th. Rel. Fields* **111**, 123-129.

Telcs, A. (1989): Random walks on graphs, electrical networks and fractals. *Probab. Th. Rel. Fields* **82**, 435-449.

Tetali, P. (1991): Random walks and the effective resistance of networks. *J. Theor. Probab.* **4**, 101-109.

Tetali, P. (1994): An extension of Foster's network theorem. *Combin. Probab. Comput.* **3**, 421-427.

Todhunter, I. (1949): *A history of the mathematical theory of probability from the time of Pascal to that of Laplace.* Chelsea, New York (first published 1865).

Urban, R. (1997): Some remarks on the random walk on finite groups. *Colloq. Math.* **74**, 287-298.

Varopoulos, N. Th. (1983a): Random walks on soluble groups. *Bull. Sci. Math.* (2) **107**, 337-344.

Varopoulos, N. Th. (1983b): Brownian motion and transient groups. *Ann. Inst. Fourier* **33**, 241-261.

Varopoulos, N. Th. (1984): Brownian motion and random walks in manifolds. *Symp. Math.* **XXIX**, 97-109.

Varopoulos, N. Th., Saloff-Coste, L. & Coulhon, T. (1992): *Analysis and geometry on groups.* Cambridge Tracts in Math. **100**, Cambridge Univ. Press.

Vilenkin, N. Ya. (1968): *Special functions and the theory of group representations.* Transl. Math. Monographs **22**, Amer. Math. Soc., Providence, RI.

Wendel, J. G. (1958): Spitzer's formula: a short proof. *Proc. Amer. Math. Soc.* **9**, 905-908.

Wendel, J. G. (1960): Order statistics of partial sums. *Ann. Math. Statist.* **31**, 1034-1044.

Wendel, J. G. (1962): Brief proof of a theorem of Baxter. *Math. Scand.* **11**, 107-108.

Wendel, J. G. (1975): Left-continuous random walk and the Lagrange expansion. *Amer. Math. Monthly* **82**, 494-499.

Whitworth, W. A. (1886): *Choice and chance*, 4th ed., Deighton Bell, Cambridge (5th ed. 1910; reprinted G. E. Stechert, New York, 1934).

Wiener, N. & Hopf, E. (1931): Über eine Klasse singuläre Integralgleichungen. *Sitzungsberichte Berliner Akad. Wiss.*, 696-706.

Williams, D. (1974): Path decomposition and continuity of local time for one-dimensional diffusions. *Proc. London Math. Soc.* (3) **28**, 738-768.

Woess, W. (1991): Behaviour at infinity of harmonic functions of random walks on graphs. *Probability on Groups X* (ed. Heyer, H.) 437-458, Plenum Press, New York & London.

Woess, W. (1994): Random walks on infinite graphs and groups - a survey on selected topics. *Bull. London Math. Soc.* **26**, 1-60.

Zemanian, A. H. (1992): *Infinite electrical networks.* Cambridge Tracts in Math. **101**, Cambridge Univ. Press.

Zolotarev, V. M. (1957): Mellin-Stieltjes transform in probability theory. *Th. Probab. Appl.* **2**, 433-460.

Zuckerman, D. (1989): Covering times of random walks on bounded degree trees and other graphs. *J. Theor. Probab.* **2**, 147-157.