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ABELIAN, TAUBERIAN AND MERCERIAN THEOREMS FOR ARITHMETIC SUMS

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§1. Introduction

Given a function $f : [2, \infty) \to [0, \infty)$, the irregularity of the sequence of primes p means that the sequence $f(p)$ will also be irregular, for reasonably regular f. To obtain reasonable regularity, one needs to sum, and consider arithmetic sums of the form

$$F(x) := \sum\nolimits_{p \le x} f(p).$$

Often divisibility properties are under study, when one needs to consider the double sums

$$\frac{1}{x} \sum\nolimits_{2 \le n \le x} \sum\nolimits_{p|n} f(p).$$

We consider Abelian results, passing from behaviour of f to that of single or double sums as above, Tauberian theorems - converse implications under additional conditions (Tauberian conditions), and Mercerian theorems, in which we pass from some comparison statement between f and a sum to a conclusion on f alone.

This line of work may be traced to a seminal paper of Pólya in 1917 ([P]; see also Pólya & Szegö [PS, Part II, Ch. 4, No. 156]). Our two main tools are the Karamata theory of regular variation (originated in 1930: see [BGT]) and the Wiener Tauberian theory (originated in 1932: see Hardy [H], Widder [W]). Pólya's achievement is all the more striking in that neither of these tools was available to him.

This study arises from a fusion of two recent lines of work. On the arithmetic sums side, our interest was stimulated by a series of studies by De Koninck & Ivić [DeKI1,2,3]. On the Abelian-Tauberian-Mercerian side, we make use of recent work of our own [BI5], itself a sequel to a series of earlier studies [BI1-4].

To take a motivating example, consider the case $f(x) \equiv 1$. Then

$$\sum\nolimits_{p|n} 1 = \omega(n),$$

the number of prime divisors of n counted without multiplicity, and the double sum above is

$$\frac{1}{x}\sum\nolimits_{2\le n\le x}\sum\nolimits_{p|n} 1 = \frac{1}{x}\sum\nolimits_{n\le x}\omega(n),$$

the *average order* of ω. This has been studied since Hardy and Ramanujan in 1917 ([HR]; [HW, Th. 430]). A refinement with a more precise error term is in Tenenbaum [T, I.5.3, Th. 6]:

$$\frac{1}{x}\sum\nolimits_{n\le x}\omega(n) = \log\log x + c_1 + O(1/\log x) \qquad (x\to\infty), \tag{1.1}$$

where $c_1 = 0.261497\ldots$ is a known constant ([HW, Th. 430]; see also [HW, Th. 427, 428]). Our methods give

$$\frac{1}{\lambda x}\sum\nolimits_{n\le\lambda x}\omega(n) - \frac{1}{x}\sum\nolimits_{n\le x}\omega(n) \sim \frac{\log\lambda}{\log x} \qquad (x\to\infty) \qquad \forall\lambda>1, \tag{1.2}$$

or equivalently (by the 'representation theorem for Π': [BGT, Th. 3.6.6])

$$\frac{1}{x}\sum\nolimits_{n\le x}\omega(n) = C + \int_2^x (1+o(1))\frac{dt}{t\log t} + o\Big(\frac{1}{\log x}\Big) \tag{1.3}$$

for some constant C.

Neither of these two results contains the other; we pause to compare them. In (1.1), which is classical in character, it is the *order of magnitude* of each of the three terms that counts. In (1.3), new in character to our knowledge, it is the *behaviour under differencing* that counts. The '1' term in the integrand gives the $\log\log x$ term in (1.1) in order of magnitude, and the main term $\log\lambda/\log x$ in (1.2) on differencing. The $o(1)$ term in the integrand gives merely $o(\log\log x)$ in order of magnitude, but $o(1/\log x)$ on differencing (by the Uniform Convergence Theorem for Π: [BGT, Th. 3.1.16]). The two error terms, both $o(1/\log x)$, may be combined, and the constant C, whose value is immaterial, goes out on differencing. Our result thus provides a new complement to (1.1), of a different character to the classical ones: the Erdös-Kac central limit theorem, and its Berry-Esseen refinement, the Rényi-Turán theorem [T, III.4.4].

Turning to single sums $F(x) := \sum_{p\le x} f(p)$, exactly the same remarks apply to the classical formula

$$\sum\nolimits_{p\le x} 1/p = \log\log x + c_1 + o(1/\log x) \tag{1.4}$$

([T, I.1.4, Th. 9]; [HW, Th. 427] with $o(1)$ error term); our methods give the alternative form

$$\sum_{p\le x} 1/p = C + \int_2^x (1+o(1))\frac{dt}{t\log t} + o\Big(\frac{1}{\log x}\Big). \tag{1.5}$$

Similar remarks apply to Mertens' first theorem, which in its modern form is

$$\sum_{p\le x} \frac{\log p}{p} = \log x + O(1) \tag{1.6}$$

([T, I.1.4, Th. 7], [HW, Th. 425]). We obtain

$$\sum_{p\le x} \frac{\log p}{p} = C + \int_1^x (1+o(1))\frac{dt}{t} + o(1). \tag{1.7}$$

The occurrence of representations like ours with two error terms, one under the integral sign and one not, is characteristic of the theory of regular variation, in both its Karamata and de Haan forms (for which see e.g. [BGT, Ch. 1,2] and [BGT, Ch. 3] respectively). It is discussed in, e.g., [B, p. 223], itself suggested by [DS]; both these studies, like the present paper, were motivated by number theory.

We now introduce the terminology needed to formulate our results. For $\rho \in \mathbb{R}$, we write R_ρ for the (Karamata) class of functions g regularly varying with index ρ: positive, measurable, and with

$$g(\lambda x)/g(x) \to \lambda^\rho \qquad (x\to\infty) \qquad \forall \lambda > 0.$$

Functions in R_0 are called slowly varying; we use ℓ for the generic slowly varying function. For $\ell \in R_0$, the (de Haan) class Π, or Π_ℓ, is the class of measurable g with

$$\{g(\lambda x) - g(x)\}/\ell(x) \to c\log\lambda \qquad (x\to\infty) \qquad \forall\lambda > 0$$

for some constant $c \in \mathbb{R}$, called the ℓ-index of g. For a kernel $k : (0,\infty) \to \mathbb{R}$, for which the Mellin convolution

$$\check{k}(s) := \int_0^\infty t^{-s}k(t)dt/t$$

converges absolutely in some strip (possibly a line) in the complex s-plane, we write

$$(f*k)(x) := \int_0^\infty k(x/t)f(t)dt/t$$

for the Mellin convolution of f and k, when absolutely convergent. We have

$$\sum_{2\le n\le x}\sum_{p|n} f(p) = \sum_{p\le x} f(p)[\frac{x}{p}], \tag{1.8}$$

where [.] denotes the integer part. This leads to the relevance of the particular kernel

$$k(x) := I_{(1,\infty)}(x).[x]/x, \tag{1.9}$$

with Mellin transform

$$\check{k}(s) = \zeta(1+s)/(1+s), \tag{1.10}$$

where $\zeta(.)$ is the Riemann zeta function (for background on which see [I], [Ti]). This kernel is that relevant to the work of Pólya cited earlier; accordingly, we call it the *Pólya kernel.*

In [DeKI1], De Koninck and Ivić showed that, for $\rho > 0$ and $\ell \in R_0$,

$$f(x) \sim x^\rho \ell(x) \qquad (x \to \infty) \tag{1.11}$$

implies

$$\sum\nolimits_{2 \le n \le x} \sum\nolimits_{p|n} f(p) \sim \frac{x^{\rho+1}\ell(x)}{\log x} . \frac{\zeta(\rho+1)}{\rho+1} \qquad (x \to \infty). \tag{1.12}$$

Note that (1.11) and (1.12) together imply the comparison statement

$$\sum\nolimits_{n \le x} \sum\nolimits_{p|n} f(p) \sim f(x). \frac{x}{\log x}.C \qquad (x \to \infty), \tag{1.13}$$

where

$$C = \frac{\zeta(\rho+1)}{(\rho+1)}. \tag{1.14}$$

In [DeKI3, Th. 4], De Koninck and Ivić showed that (1.13) with $C > 0$ implies (1.11), so also (1.12), for some $\rho > 0$ and $\ell \in R_0$, and then C is the unique solution of (1.14). They called this result 'Tauberian'. We take a different view: we call results of the form (1.11) $\Rightarrow$ (1.12) *Abelian*, those of the form (1.12) $\Rightarrow$ (1.11) under additional conditions *Tauberian*, the conditions (on f) being *Tauberian conditions*, and those of the form (1.13) $\Rightarrow$ (1.11) *Mercerian.* For background to this terminology, see [BGT, Ch. 4,5] and [BI1-5]. For single sums $\sum_{p \le x} f(p)$, it will be seen that the (Mercerian) 'comparison constant' is $1/(1+\rho)$, with general (Karamata) case $\rho > -1$ and boundary (de Haan) case $\rho = -1$, while for double sums $\sum_{n \le x} \sum_{p|n} f(p)$, the comparison constant is $\zeta(1+\rho)/(1+\rho)$, with general case $\rho > 0$ and boundary case $\rho = 0$.

Our agenda in this paper is the following:

(i) establishing our differencing results for the arithmetic functions above;

(ii) the Tauberian implication (1.12) $\Rightarrow$ (1.11) for the case $\rho > 0$ under the Tauberian condition that f is non-decreasing (Theorem 3.1);

(iii) the corresponding Abelian and Tauberian theorems for the case $\rho = 0$ (Theorem 5.1,

containing (1.2), (1.3) above, and Theorem 5.3);
(iv) a Tauberian theorem of de Haan type based on Wiener Tauberian theory (Theorem 4.2), rather than Korenblum theory as in [BI5, Th. 4.1];
(v) an Abelian theorem for the single sums $\sum_{p\le x} f(p)$, Theorem 2.1(i) (containing (1.5), (1.7) above), its Tauberian counterpart, Theorem 2.1(ii), and Mercerian results, Theorems 2.2 ($\rho > -1$) and 2.4 ($\rho = -1$);
(vi) a simplified and extended form of the Mercerian theorem of De Koninck & Ivić [DeKI3, Th. 4], Theorem 3.2;
(vii) a special-kernel Mercerian theorem, Theorem 2.3, of independent interest. This provides a complement to de Haan's theorem [BGT, Th. 3.7.3].

In §2 below we prove Abelian, Tauberian and Mercerian theorems for the single sums $\sum_{p\le x} f(p)$. Writing

$$\pi(x) := \sum\nolimits_{p\le x} 1$$

for the simplest and most important of such sums, we use the Prime Number Theorem (PNT) in the form

$$\pi(x) = \int_2^x \frac{dt}{\log t} + R(x) \qquad (2 \le x < \infty), \tag{1.15}$$

where, as in [DeKI1,3]

$$R(x) = O(xe^{-\sqrt{\log x}}) \qquad (x \to \infty). \tag{1.16}$$

(The error term $O(xe^{-c\sqrt{\log x}})$ for some $c > 0$, used in [T, II.4.1], would also do; for the best error term known, see [I, Ch. 12]).

In §3 we turn to the double sum $\frac{1}{x}\sum_{n\le x}\sum_{p|n} f(p)$. Our method is to reduce (1.12) to

$$\int_2^x \frac{f(t)}{\log t}[\frac{x}{t}]dt \sim \frac{x^{\rho+1}\ell(x)}{\log x}.\frac{\zeta(\rho+1)}{(\rho+1)}. \tag{1.17}$$

Writing

$$\tilde{f}(x) := I_{[2,\infty)}(x)f(x)/\log x, \qquad \tilde{\ell}(x) := \ell(x)/\log x, \tag{1.18}$$

one rewrites (1.17) in convolution form as

$$\frac{1}{x}\int_2^x \frac{f(t)}{\log t}[\frac{x}{t}]dt = (k * \tilde{f})(x) \sim x^\rho \tilde{\ell}(x)\check{k}(\rho) \qquad (x \to \infty), \tag{1.19}$$

with k the Pólya kernel as above. The Tauberian theorem [BI5, Th. 3.1], based on Korenblum's form of the Wiener Tauberian theorem, then allows one to pass from (1.19) to (1.11), under the Tauberian condition that f be (for simplicity) non-decreasing. In the Tauberian result Theorem 3.1, we pass from (1.12) for $\rho > 0$ to (1.11) by this route, following this with its Mercerian counterpart, Theorem 3.2. In §4, we give a Tauberian theorem for Π-variation for systems of kernels (see [BI5] for the systems aspect), for use in §5.

Finally, in §5 we consider the case $\rho = 0$. We prove the Abelian result (Theorem 5.1) that

$$f(x) \sim \ell(x) \qquad (x \to \infty) \qquad (\ell \in R_0) \tag{1.20}$$

implies

$$\frac{1}{x}\sum\nolimits_{n \le x}\sum\nolimits_{p|n} f(p) \in \Pi_{\tilde{\ell}} \qquad \text{with } \tilde{\ell}\text{-index } 1; \tag{1.21}$$

note that $f(x) \equiv 1$ gives $\sum_{p|n} 1 = \omega(n)$, so this gives (1.2), (1.3). We also prove the Tauberian counterpart, Theorem 5.2, passing from (1.21) to (1.20) under the Tauberian condition that f be (for simplicity) non-increasing. We find it necessary to restrict ℓ here, by

$$\int^{\infty} \ell(x) e^{-\sqrt{\log x}} dx/x < \infty \tag{1.22}$$

and

$$\log x = O(\ell(x)) \tag{1.23}$$

(here and below, '$x \to \infty$' will be understood in limits and asymptotic statements). Now (1.22) is not too restrictive, being satisfied by $\ell(x) = 1$, $\ell(x) = \log x$, and other familiar examples (though it fails for $\ell(x) = \exp(\sqrt{\log x})$ and other such 'rapidly growing' slowly varying functions). Much more serious is (1.23), which fails for $\ell(x) \equiv 1$, although this case is relevant to (1.1)-(1.3), one of our motivating examples. We suspect that (1.23) and possibly (1.22) are inessential, but it may be that such restrictions reflect the greater delicacy of the case $\rho = 0$. Our method of proof (the obvious and standard one: see e.g. [T], [DeKI1-3]) uses integration by parts to reduce (1.21), written in 'convolution sum' form by (1.8), to the convolution integral form

$$(k * \tilde{f}) \in \Pi_{\tilde{\ell}} \qquad \text{with } \tilde{\ell}\text{-index } 1, \tag{1.24}$$

and it is here that (1.22), (1.23) are needed (see §5, (5.6) and Step 4 of the proof).

We note that an extension of the work of §5 to include the case $\ell(x) \sim 1$, if possible, would provide an 'asymptotic arithmetic characterization of the identity function', to adapt the terminology of De Koninck *et al.* [DeKKP].

Tauberian implications of the form (1.24) $\Rightarrow$ (1.20) are considered in [BI5, Th. 4.1], using Korenblum's theorem. However, this result is not applicable here, because of restrictions on our knowledge of the zero-free region of the Riemann zeta-function. The best zero-free region known is of the form

$$\sigma \geq 1 - C(\log t)^{-2/3}(\log\log t)^{-1/3} \qquad (t \geq t_0) \qquad (s = \sigma + it)$$

for some constants C and t_0 [I, Ch. 6]. The method of [BI5] would require a zero-free region of $\check{k}$ in (1.10) of the form $-\epsilon < \Re s < \epsilon$ for some $\epsilon > 0$, that is, a half-plane $\sigma \geq 1 - \epsilon$ for $\zeta(s)$, far beyond what is known. Of course, the error term in PNT and the zero-free region of $\zeta(s)$ are intimately linked; for quantitative results here, see [I, Th. 12.3].

§2. Single sums

For completeness, we begin with

Theorem 2.0. *(i) For $\rho > -1$, (1.11) implies*

$$\sum\nolimits_{p \leq x} f(p) \sim \frac{x^{\rho+1}\ell(x)}{\log x}.\frac{1}{(\rho+1)} \qquad (x \to \infty). \tag{2.1}$$

(ii) Conversely, under the Tauberian condition that f is monotone, (2.1) implies (1.11).

Proof. (i) The Abelian assertion is proved in [DeKI2].
(ii) The converse follows as in the proof of the Monotone Density Theorem [BGT, Th. 1.7.2], or of Theorem 2.1 below; we omit the details.

We turn now to the limiting case $\rho = -1$, motivated by cases such as (1.4)-(1.7). The Tauberian content of the second part is a classical monotonicity argument, as in e.g. [BGT, §§1.7.3, 3.6.5].

Theorem 2.1. *(i) For $f : [2, \infty)$, $\ell \in R_0$, $\tilde{\ell}(x) := \ell(x)/\log x$,*

$$f(x) \sim \ell(x)/x \qquad (x \to \infty) \tag{2.2}$$

implies

$$\sum\nolimits_{p \leq x} f(p) \in \Pi_{\tilde{\ell}} \qquad \textit{with } \tilde{\ell}\textit{-index 1.} \tag{2.3}$$

(ii) Conversely, if f is non-increasing, (2.3) implies (2.2).

Proof. (i) We first prove the Abelian assertion (2.2) $\Rightarrow$ (2.3). Since the right of (2.2) is measurable, we may assume without loss that the left, f, is measurable.

Take $\lambda > 1$. For g Riemann integrable on $[1, \lambda]$, we can prove

$$\frac{\log x}{x} \sum\nolimits_{x<p\leq \lambda x} g(p/x) \to \int_1^\lambda g(t)dt,$$

as in the proof of Pólya's theorem: see [P], [PS, Part II, Ch. 4, No. 156], [BGT, §6.4.4], or the proof of Lemma 5.2 below. In particular,

$$\log x \sum\nolimits_{x<p\leq \lambda x} 1/p \to \int_1^\lambda dt/t = \log \lambda. \tag{2.4}$$

By the Uniform Convergence Theorem [BGT, Th. 1.2.1], for any $\epsilon > 0$, (2.2) gives $M = M(\epsilon)$ with

$$1 - \epsilon \leq \frac{pf(p)}{xf(x)} \leq 1 + \epsilon \qquad (M \leq x \leq p \leq \lambda x).$$

So with $F(x) := \sum_{p\leq x} f(p)$, $F(\lambda x) - F(x)$ is bounded between $(1 \pm \epsilon) x f(x) \sum_{x<p\leq \lambda x} 1/p$ for $x \geq M$. By (2.4), the sum here is asymptotic to $\log \lambda / \log x$, and by (2.2) $xf(x)/\log x \sim \ell(x)/\log x = \tilde{\ell}(x)$. So for large enough x, $F(\lambda x) - F(x)$ is bounded between $(1 \pm 2\epsilon)\tilde{\ell}(x)$. Taking upper and lower limits of $\{F(\lambda x) - F(x)\}/\tilde{\ell}(x)$ yields (2.3).
(ii) For the Tauberian part, assume (2.3) and f non-increasing. By (2.3),

$$\sum\nolimits_{x<p\leq \lambda x} f(p) \sim \tilde{\ell}(x) \log \lambda = \frac{\ell(x)}{\log x}.\log \lambda \qquad \forall \lambda > 1.$$

The monotonicity of f now yields

$$\sum\nolimits_{x<p\leq \lambda x} f(p) \leq f(x)\{\pi(\lambda x) - \pi(x)\} \sim \frac{xf(x)}{\log x}(\lambda - 1)$$

(using $\pi(x) \sim x/\log x$, the PNT without remainder). Combining,

$$\liminf\nolimits_{x\to\infty} \frac{xf(x)}{\ell(x)} \geq \frac{\log \lambda}{(\lambda - 1)},$$

and letting $\lambda \downarrow 1$,

$$\liminf\nolimits_{x\to\infty} \frac{xf(x)}{\ell(x)} \geq 1.$$

Similarly, the limsup is ≤ 1, and (2.2) follows.

For $\ell \in R_0$, ℓ is locally bounded on $[M, \infty)$ for M large enough [BGT, Lemma 1.3.2]. Then

$$L(x) := \int_M^x \tilde{\ell}(x) \frac{dt}{t} = \int_M^x \ell(x) \frac{dt}{t \log t} \tag{2.5}$$

defines another slowly varying function, with

$$L(x)/\tilde{\ell}(x) = L(x)\log x/\ell(x) \to \infty,$$

by [BGT, Prop. 1.5.9] (for further background on the link between L and ℓ, see [BGT §3.7, p. 162, 164]). Thus $\ell \equiv 1$ gives $L(x) \sim \log\log x$, while $\ell(x) = \log x$ gives $L(x) \sim \log x$. The 'second-order' statement (2.3), in terms of ℓ or $\tilde{\ell}$, thus gives in particular a cruder 'first-order' statement in terms of L.

COROLLARY. *For ℓ such that $\int^{\infty} \ell(t)dt/t\log t = \infty$, (2.2) implies*

$$\sum\nolimits_{p\le x} f(p) \sim L(x).$$

Example 2.1. (i) Taking $\ell \equiv 1$ in the Corollary gives

$$\sum\nolimits_{p\le x} 1/p \sim \log\log x, \tag{2.6}$$

which is (1.4) without constant or remainder term.
(ii) Taking $\ell(x) \sim \log x$ yields

$$\sum\nolimits_{p\le x} \frac{\log p}{p} \sim \log x, \tag{2.7}$$

which is (1.6) without remainder term.

We now turn to the Mercerian aspects, beginning with the case $\rho > -1$. This uses Karamata's theorem [BGT, Th. 1.6.1], the prototype of Mercerian theorems in regular variation.

THEOREM 2.2. *Let $f : [2,\infty) \to [0,\infty)$ be left-continuous and monotone (non-decreasing or non-increasing). If*

$$\sum\nolimits_{p\le x} f(p) \sim C.\frac{x}{\log x}.f(x) \tag{2.8}$$

for some $C \in (0,\infty)$, then $f \in R_\rho$ with $\rho = C^{-1} - 1$.

Proof. First note that we must have $\sum_p f(p) = \infty$. For $\sum_p f(p) < \infty$ would give $f(x) \sim$ *const.* $\log x/x$, and then (2.7) would contradict $\sum_p f(p) < \infty$. Thus $\sum_p f(p) = \infty$, and so by (2.8),

$$xf(x)/\log x \to \infty. \tag{2.9}$$

Now

$$\sum\nolimits_{p\leq x} f(p) = \int_{[2,x]} f(t)d\pi(t),$$

with π the prime-counting function. So by (1.15),

$$\sum\nolimits_{p\leq x} f(p) = \int_2^x \frac{f(t)}{\log t}dt + \int_{[2,x]} f(t)dR(t).$$

Now π, and so R, is right-continuous. Setting $R(x) = 0$, $f(x) = f(2)$ for $x < 2$, we may use integration by parts ((A1) in the Appendix, with R as f and f as g) to write

$$\int_{[2,x]} f(t)dR(t) = f(x)R(x) + O(1) - \int_{[2,x)} R(t)df(t).$$

The weak form $R(x) = O(x/e^{\sqrt{\log x}}) = o(x/\log x)$ and (2.9) show that the integrated terms are $o(xf(x)/\log x)$. Now

$$\begin{aligned}\int_{[2,x)} R(t)df(t) &<< \int_{[2,x)} te^{-\sqrt{\log t}}df(t) \\ &= xe^{-\sqrt{\log x}}f(x) + O(1) + \int_2^x f(t)\{\frac{1}{2\sqrt{\log t}} - 1\}.e^{-\sqrt{\log t}}dt.\end{aligned}$$

Again, the integrated terms are $o(xf(x)/\log x)$. The integral is

$$<< \int_2^x \frac{f(t)}{\log t}dt.$$

If $\int_2^\infty f(t)dt/\log t$ converged, (2.9) would give $\int_2^x f(t)dt/\log t = o(xf(x)/\log x)$. This and the argument above would give $\sum_{p\leq x} f(p) = o(xf(x)/\log x)$. But this contradicts our Mercerian assumption (2.8), and so

$$\int_2^\infty \frac{f(t)}{\log t}dt = \infty.$$

Thus behaviour at infinity dominates, and so

$$\int_2^x \{\frac{1}{2\sqrt{\log t}} - 1\}e^{-\sqrt{\log t}}f(t)dt = o(\int_2^x \frac{f(t)}{\log t}dt).$$

Combining,

$$\sum\nolimits_{p\leq x} f(p) = \{1 + o(1)\}\int_2^x \frac{f(t)}{\log t}dt + o(\frac{xf(x)}{\log x}).$$

This and our Mercerian assumption (2.8) give

$$\int_2^x \frac{f(t)}{\log t} dt \sim C.\frac{xf(x)}{\log x}.$$

By Karamata's theorem, $f(x)/\log x \in R_\rho$ with $C = 1/(1+\rho)$. That is, $f \in R_\rho$ with $\rho = C^{-1} - 1$, as required.

We turn now to the Mercerian theorem in the limiting case $\rho = -1$. First, recall the Matuszewska indices of a positive function [BGT, §2.1.2]. The *upper Matuszewska index* $\alpha(f)$ is the infimum of those α for which there exists a constant $C = C(\alpha)$ for which, for each $\Lambda > 1$,

$$f(\lambda x)/f(x) \le C\{1 + o(1)\}\lambda^\alpha \qquad (x \to \infty) \qquad \text{uniformly in } \lambda \in [1, \Lambda];$$

dually, the *lower Matuszewska index* is the supremum of those β for which there exists a constant $D = D(\beta)$ such that for each $\Lambda > 1$,

$$f(\lambda x)/f(x) \ge \{1 + o(1)\}\lambda^\beta \qquad (x \to \infty) \qquad \text{uniformly in } \lambda \in [1, \Lambda].$$

One says that f has *bounded increase*, $f \in BI$, if $\alpha(f) < \infty$, *bounded decrease*, $f \in BD$, if $\beta(f) > -\infty$.

Theorem 2.3 below is a Mercerian theorem, complementary to de Haan's theorem [BGT, Th. 3.7.3, (3.7.12)]. Its Mercerian content is [BGT, Th. 5.2.3], a form of the Drasin-Shea theorem, the principal Mercerian theorem for non-negative kernels.

THEOREM 2.3. *Let g be non-negative and measurable on $(0, \infty)$, with $g \in BD \cup BI$. If*

$$\int_x^{\lambda x} g(t)dt/t \sim g(x) \log \lambda \qquad (x \to \infty) \tag{2.10}$$

for some $\lambda > 1$, then $g \in R_0$.

Proof. Writing $k(x) := I_{(\lambda^{-1},1)}(x)$, $\int_x^{\lambda x} g(t)dt/t = (k * g)(x)$ in convolution form. Then (2.10) is

$$(k * g)(x)/g(x) \to \log \lambda.$$

The Mellin convolution is

$$\check{k}(s) = (\lambda^s - 1)/s \qquad (s \neq 0), \qquad \log \lambda \qquad (s = 0),$$

the strip of convergence being the whole complex s-plane $\mathbb{C}$. By [BGT Th. 5.2.3], $\check{k}(\rho) = \log\lambda$ for some $\rho \in \mathbb{R}$, and $g \in R_\rho$. The functional form of $\check{k}$ above forces $\rho = 0$, and so $g \in R_0$ as required.

Remark. Theorem 2.3, a special-kernel Mercerian theorem involving one λ, may be compared with [BI5, Th. 6.1], a general kernel Mercerian theorem involving two (logarithmically incommensurable) λs.

The extension of Theorem 2.2 to the boundary case $\rho = -1$ has a Mercerian hypothesis involving differencing. Its Mercerian content is Theorem 2.3.

THEOREM 2.4. *If $f : [2,\infty) \to [0,\infty)$ is left-continuous, non-increasing, and*

$$\sum\nolimits_{x<p\leq\lambda x} f(p) \sim \frac{xf(x)}{\log x}.\log\lambda \tag{2.11}$$

for some $\lambda > 1$, then $f \in R_{-1}$.

Proof. Integrating by parts as before,

$$\sum\nolimits_{x<p\leq\lambda x} f(p) = \int_{[x,\lambda x]} f(t)d\pi(t) = \int_x^{\lambda x} \frac{f(t)}{\log t}dt + \int_{[x,\lambda x]} f(t)dR(t).$$

Integrate the second integral by parts: as before, the integrated terms are $o(xf(x)/\log x)$. Applying (A1) as before, we obtain

$$\int_{[x,\lambda x)} R(t)df(t) = o\Big(\frac{xf(x)}{\log x}\Big).$$

So (2.11) says that

$$\int_x^{\lambda x} \frac{f(t)}{\log t}dt \sim \frac{xf(x)}{\log x}.\log\lambda.$$

Since f non-increasing implies $xf(x)/\log x \in BI$, Theorem 2.3 gives $xf(x)/\log x \in R_0$, or $f \in R_{-1}$.

§3. Double sums: $\rho > 0$

Recall the Abelian result of [DeKI1]: (1.11) $\Rightarrow$ (1.12). We begin with the Tauberian converse. Its Tauberian content is [BI5, Th. 3.1].

THEOREM 3.1. *For $\rho > 0$ and f positive and non-decreasing, (1.12) implies (1.11).*

Proof. Given f non-decreasing, we can find g continuous and non-decreasing agreeing with f on the positive integers. Since (1.11) for g implies (1.11) for f, we may (and shall) take

f continuous.

Step 1. By non-decrease of f,

$$\sum_{p\le 2x} f(p)[\frac{2x}{p}] \ge \sum_{x\le p\le 2x} f(p) \ge f(x)\{\pi(2x)-\pi(x)\}.$$

By (1.8) and (1.12), the left is $O(x^{\rho+1}\ell(x)/\log x)$, while by PNT the second factor on the right is $\sim x/\log x$. So

$$f(x) << x^{\rho}\ell(x). \tag{3.1}$$

On the other hand,

$$\sum_{p\le x} f(p)[\frac{x}{p}] \le xf(x)\sum_{p\le x} 1/p \sim xf(x)\log\log x,$$

by (2.6). So (1.12) gives

$$f(x) >> \frac{x^{\rho}\ell(x)}{\log x\log\log x}. \tag{3.2}$$

We claim

$$\sum_{p\le x} f(p)[\frac{x}{p}] = \{1+o(1)\}\int_2^x \frac{f(t)}{\log t}[\frac{x}{t}]dt + o(x^{\rho+1}\ell(x)). \tag{3.3}$$

Now

$$\begin{aligned}\sum_{p\le x} f(p)[\frac{x}{p}] &= \int_2^x f(t)d\pi(t)\\ &= \int_2^x \frac{f(t)}{\log t}[\frac{x}{t}]dt + \int_{[2,x]} f(t)[\frac{x}{t}]dR(t).\end{aligned} \tag{3.4}$$

We integrate the second integral on the right by parts. The integrated term is zero (as $[x/x+]=0$ and $R(2-)=0$); the integral term is (the negative of)

$$\int_{[2,x]} R(t)d\{f(t)[\frac{x}{t}]\} = \int_2^x R(t)[\frac{x}{t}]df(t) + \int_{[2,x]} R(t)f(t)d[\frac{x}{t}],$$

by Theorem B of the Appendix. Since

$$\frac{x}{t} \le [\frac{x}{t}]+1 \le 2[\frac{x}{t}] \qquad (2\le t\le x),$$

(1.16) and (3.1) give, on integrating by parts,

$$\begin{aligned}\int_2^x R(t)[\frac{x}{t}]df(t) &<< x\{f(x)e^{-\sqrt{\log x}} + O(1) + \frac{1}{2}\int_2^x \frac{f(t)}{t\sqrt{\log t}}e^{-\sqrt{\log t}}dt\}\\ &<< o(x^{\rho+1}\ell(x)/\log x) + \int_2^x [\frac{x}{t}]\frac{f(t)}{\log t}\{\sqrt{\log t}e^{-\sqrt{\log t}}\}dt\\ &= o(x^{\rho+1}\ell(x)/\log x) + o(\int_2^x \frac{f(t)}{\log t}[\frac{x}{t}]dt)\end{aligned}$$

(recall the bounds (3.1) and (3.2)). On the other hand, arguing as in the proof of [DeKI3, Lemma 2],

$$\int_{[2,x]} R(t)f(t)d[\frac{x}{t}] = -\sum\nolimits_{n \leq x/2} f(x/n)R(x/n) << xf(x)e^{-C\sqrt{\log x}}$$

for some $C > 0$, by (3.2), or $o(x^{\rho+1}\ell(x)/\log x)$ by (3.1). Combining, we obtain (3.3).
Step 2. Using (3.3), our assumption (1.12) is

$$(k * \tilde{f})(x) \sim x^{\rho}\tilde{\ell}(x)\check{k}(\rho), \tag{3.5}$$

using the notation of (1.18), with k the Pólya kernel (so $\check{k}(s) = \zeta(1+s)/(1+s)$). Now with μ the Möbius function, for $s = \sigma + it$ with $\sigma > 1$ we have

$$|\zeta(s)| \leq \sum\nolimits_{n=1}^{\infty} |1/n^s| = \sum\nolimits_{1}^{\infty} 1/n^{\sigma} = \zeta(\sigma),$$

while

$$|1/\zeta(s)| = |\sum_{1}^{\infty} \mu(n)/n^s| \leq \sum_{1}^{\infty} 1/n^{\sigma} = \zeta(s)$$

([T, (1.1.4)], or [I, (1.84)]. Combining,

$$1/\zeta(\sigma) \leq |\zeta(s)| \leq \zeta(\sigma).$$

So for $\sigma > 0$,

$$|\log|\zeta(1+\sigma+it)|| \leq \log|\zeta(1+\sigma)|,$$

uniformly in t, while $\log|(1+\sigma+it)| = O(\log t)$ as $t \to \pm\infty$. So for $\sigma > 0$ and $\epsilon > 0$, $\check{k}(s) = \zeta(1+s)/(1+s)$ satisfies the Nyman-Korenblum decay condition

$$\frac{\log|\check{k}(\sigma+it)}{\exp\left(\frac{\pi|t|}{2\epsilon}\right)} \to 0 \qquad (t \to \pm\infty)$$

(with much to spare); see [BI5] for background and references here. Now the Tauberian condition that f is non-decreasing gives $x\tilde{f}(x) = xf(x)/\log x$ eventually non-decreasing, whence $\log \tilde{f}(x)$ is slowly decreasing [BGT, §1.7.6]. Now [BI5, Th. 3.1] applies, and from (3.5) we obtain

$$\tilde{f}(x) \sim x^{\rho}\tilde{\ell}(x),$$

or $f(x) \sim x^{\rho}\ell(x)$, which is (1.11) as required.

We finish this section with the Mercerian complement to the result above. This extends and simplifies [DeKI3, Th. 4], the result that motivated this study. The Mercerian content is the Drasin-Shea theorem [BGT, Th. 5.2.1].

THEOREM 3.2. *If $f : [2, \infty) \to [0, \infty)$ is continuous and non-decreasing, and*

$$\sum\nolimits_{2 \le n \le x} \sum\nolimits_{p|n} f(p) \sim C.\frac{x}{\log x}.f(x)$$

for some constant $C \in (0, \infty)$, then $f \in R_\rho$, where ρ is the unique positive solution to $C = \zeta(1+\rho)/(1+\rho)$.

Proof. First, since $\int_2^\infty dt/t \log t$ diverges and f is non-decreasing, $\int_2^\infty f(t) dt/t \log t$ diverges also.

As in [DeKI3, Lemma 1], set

$$S_1(x) := x \sum\nolimits_{p \le x} f(p)/p \qquad (x \ge 2).$$

Then, by [DeKI3, (4.6)], for $\epsilon > 0$ small enough there exists $X = X(\epsilon)$ with

$$(C - \epsilon)\frac{xf(x)}{\log x} \le S_1(x) \le (1+\epsilon)(1+C)\frac{xf(x)}{\log x} \qquad (x \ge X). \tag{3.6}$$

As before,

$$\frac{S_1(x)}{x} = \int_2^x \frac{f(t)}{t \log t} dt + \int_{[2,x]} \frac{f(t)}{t} dR(t).$$

We integrate the second integral by parts, to give

$$\frac{f(x)R(x)}{x} + O(1) - \int_2^x R(t) d\{\frac{f(t)}{t}\}.$$

Arguing as before, and using (3.6),

$$\frac{f(x)R(x)}{x} << f(x)e^{-\sqrt{\log x}} = o\Big(\frac{f(x)}{\log x}\Big) = o\Big(\frac{S_1(x)}{x}\Big).$$

The $O(1)$ term is $o(\int_2^x f(t)dt/t \log t)$, as the integral diverges as $x \to \infty$. The integral gives another $f(x)R(x)/x$ term, handled as above, and two integral terms. These are

$$\begin{aligned}
\int_2^x \frac{R(t)}{t} df(t) &<< \int_2^x e^{-\sqrt{\log t}} df(t) \\
&= f(x)e^{-\sqrt{\log x}} + O(1) + \frac{1}{2}\int_2^x \frac{f(t)}{t \log t}\{\sqrt{\log t}e^{-\sqrt{\log t}}\}dt \\
&= o\Big(\frac{S_1(x)}{x}\Big) + o\Big(\int_2^x \frac{f(t)}{t \log t} dt\Big),
\end{aligned}$$

as above, and

$$\int_2^x \frac{R(t)f(t)}{t^2} dt << \int_2^x \frac{f(t)}{t} e^{-\sqrt{\log t}} dt = o\Big(\int_2^x \frac{f(t)}{t \log t} dt\Big).$$

Combining, we obtain
$$\frac{S_1(x)}{x} \sim \int_2^x \frac{f(t)}{t\log t}dt.$$

Arguing as in the proof of [DeKI3, Lemma 1], it follows from this that there exist a, b $(0 < a < b < \infty)$ such that, for X_1 large enough,

$$x^a \leq f(x) \leq x^b \qquad (x \geq X_1). \tag{3.7}$$

Hence, as in the proof of [DEKI, Lemma 2] but with df in place of $f'(t)dt$,

$$\sum\nolimits_{p\leq x} f(p)[\frac{x}{p}] \sim \int_2^x \frac{f(t)}{\log t}[\frac{x}{t}]dt,$$

and so our Mercerian assumption is

$$\int_2^x \frac{f(t)}{\log t}[\frac{x}{t}]dt \sim C\frac{xf(x)}{\log x},$$

or using again the Pólya kernel k and notation of §1,

$$(k * \tilde{f})(x) \sim C\tilde{f}(x). \tag{3.8}$$

Now (3.7) implies that the upper order $\rho := \rho(f)$ of f, where

$$\rho(f) := \limsup\nolimits_{x\to\infty} \frac{\log f(x)}{\log x},$$

which equals $\rho(\tilde{f})$ also, lies in $[a, b]$. Now f is non-decreasing, so $x\tilde{f}(x) = xf(x)/\log x$ is eventually non-decreasing, so $\tilde{f} \in BD$. By the Drasin-Shea theorem, (3.8) implies $\tilde{f} \in R_\rho$ with $C = \zeta(1+\rho)/(1+\rho)$. So $f \in R_\rho$ also. The root ρ is unique, since $r \mapsto \zeta(1+r)/(1+r)$ is strictly decreasing on $(0, \infty)$ [DeKI3, Lemma 5].

§4. A Wiener Tauberian theorem for Π-variation

In this section we prove a Wiener (rather than a Korenblum) Tauberian theorem for Π-variation, which we need in the last section.

We first prove the following form of the Wiener-Pitt Tauberian theorem for systems of kernels.

THEOREM 4.1. *Let $c \in \mathbb{R}$, k_μ $(\mu \in M)$ be a system of $L^1((0,\infty), dt/t)$-class kernels. We assume that the Mellin transforms $\check{k}_\mu(z)$ $(\mu \in M)$ have no common zeros on the line $\Re z = 0$. Let $f : (0, \infty) \to \mathbb{R}$ be bounded, measurable, and slowly decreasing. If*

$$(k_\mu * f)(x) \to c \qquad (x \to \infty) \qquad \forall \mu \in M, \tag{4.1}$$

then

$$f(x) \to c \qquad (x \to \infty). \tag{4.2}$$

Proof. The proof is almost the same as the case for M a singleton (for which see e.g. Folland [Fo, Th. (4.72)]). The set $\mathcal{L}$ of all $h \in L^1 := L^1((0,\infty), dt/t)$ satisfying

$$(h * f)(x) \to c\check{h}(0) \tag{4.3}$$

is a closed, translation-invariant, linear subspace of L^1. So for $h \in \mathcal{L}$ and $y \in (0,\infty)$, $h(y^{-1}.) \in \mathcal{L}$, so $\mathcal{L}$ is a closed ideal of L^1 [Fo, Th. (2.43)]. Since $\mathcal{L}$ includes $\{k_\mu : \mu \in M\}$ by assumption, it follows from Wiener's theorem [Fo, (4.63)] that $\mathcal{L} = L^1$. This implies that (4.3) holds for every $h \in L^1$. In particular, taking $h(x) := x^{-1} I_{(1,\infty)}(x)$, we find that

$$\frac{1}{x}\int_0^x f(t)dt \to c.$$

Since f is slowly decreasing, (4.2) follows (see e.g. [BGT, Th. 1.7.5]).

Here is our Tauberian theorem for Π-variation. Its rather complicated formulation is dictated by the needs of §5 below. For a substantial application, see Example 4.1 below.

THEOREM 4.2. *Let $\ell \in R_0$ and $c \in \mathbb{R}\backslash\{0\}$, $-\infty < \sigma_1 < \rho < \sigma_2 < \infty$, and $k : (0,\infty) \to \mathbb{R}$ be a measurable function whose Mellin transform $\check{k}(z)$ is absolutely convergent in the strip $\rho < \Re z < \sigma_2$. For $\mu, x \in (0,\infty)$, set*

$$k_\mu(x) := (\mu x)^{-\rho} k(\mu x) - x^{-\rho} k(x),$$

and assume

(k1) there exist $\mu_1, \mu_2 \in (0,\infty)\backslash\{1\}$ such that $(\log \mu_1)/(\log \mu_2)$ is irrational and $k_{\mu_i}(x) \geq 0$ for $x > 0$ and $i = 1, 2$;

(k2) $\check{k}(z)$ has a holomorphic continuation to $\sigma_1 < \Re z < \sigma_2$, except for a simple pole at $z = \rho$ with residue c;

(k3) $\check{k}(z)$ has no zeros on $\Re z = \rho$.

Let $f : [0,\infty) \to [0,\infty)$ be non-negative, measurable, locally bounded, vanish in a neighbourhood of zero, and satisfy the Tauberian condition

$$\lim_{\lambda\downarrow 1} \liminf_{x\to\infty} \inf_{t\in[1,\Lambda]} \frac{(tx)^{-\rho} f(tx) - x^{-\rho} f(x)}{\ell(x)} \geq 0 \tag{T}$$

(so $= 0$). Then

$$x^{-\rho}(k * f)(x) \in \Pi_\ell \qquad \text{with } \ell\text{-index } c \tag{4.4}$$

implies (1.11).

Proof. For $\mu \in M$, $0 < \Re z < \sigma_2 - \rho$, we have

$$\begin{aligned}\check{k}_\mu(z) &= \int_0^\infty t^{-z}\{(\mu t)^{-\rho}k(\mu t) - t^{-\rho}k(t)\}dt/t \\ &= (\mu^z - 1)\check{k}(z+\rho).\end{aligned}$$

By (k2), $\check{k}_\mu(z)$ has a holomorphic extension to the strip $\sigma_1 - \rho < \Re z < \sigma_2 - \rho$, and satisfies

$$\check{k}_\mu(0) = \lim_{z\to 0}\frac{(\mu^z - 1)}{z}.z\check{k}(z+\rho) = c\log\mu. \tag{4.5}$$

So by (k1), the integrals $\check{k}_{\mu_i}(z) := \int_0^\infty t^{-z}k_{\mu_i}(t)dt/t$ $(i = 1,2)$ converge absolutely in $\sigma_1 - \rho < \Re z < \sigma_2 - \rho$. For otherwise the Vivanti-Pringsheim theorem (Doetsch [D, Ch. 4, §5]) would give a singularity in this region, a contradiction. Also, $\check{k}_{\mu_1}(z)$ $(i = 1,2)$ have no common zeros on $\Re z = 0$. For by (k3), the possible common zeros belong to

$$\{\frac{2\pi in}{\log\mu_1} : n \in \mathbb{Z}\} \cap \{\frac{2\pi im}{\log\mu_2} : m \in \mathbb{Z}\},$$

which is equal to $\{0\}$ by (k1), while $z = 0$ is not a zero, by (4.5).

Now, (4.4) implies that, for $i = 1,2$,

$$(k_{\mu_i} * \tilde{f})(x) \sim \ell(x)\check{k}_{\mu_i}(0) \qquad (x \to \infty),$$

where

$$\tilde{f}(x) := x^{-\rho}f(x) \qquad (x > 0).$$

By [BGT, Th. 4.6.5], we have

$$\tilde{f}(x) = O(\ell(x)).$$

We may assume that ℓ and $1/\ell$ are bounded on each $(0,a]$. If we set

$$g(x) := \tilde{f}(x)/\ell(x) \qquad (0 < x < \infty),$$

then g is bounded on $(0,\infty)$. As in the proof of [BGT, Th. 4.8.3], it follows that

$$(k_{\mu_i} * g)(x) \to \check{k}_{\mu_i} \qquad (i = 1,2),$$

and that g is slowly decreasing. So by Theorem 4.1, we obtain

$$g(x) \to 1,$$

or (1.11), as required.

Example 4.1. The following kernel - the *Kohlbecker kernel* - occurs in the theory of partitions in number theory [BGT, §6.1]:

$$k(x) := \frac{1}{(xe^{1/x} - 1)} \qquad (x > 0).$$

The Mellin transform is

$$\check{k}(z) = \Gamma(1+z)\zeta(1+z) \qquad (\Re z > 0)$$

(cf. [BGT, p. 233]). So $\check{k}$ has an analytic continuation to $\Re z > -1$, with a unique singularity at $z = 0$, which is a simple pole with residue 1. Also, $\check{k}(z)$ has no zeros on the line $\Re z = 0$. Since k is increasing on $(0, \infty)$, (k1) holds with $\rho = 0$ and $\mu_1 = 2$, $\mu_2 = 3$ say. So if f is as in Theorem 4.2 with $\rho = 0$, then $k * f \in \Pi_\ell$ with ℓ-index 1 implies $f(x) \sim \ell(x)$ as $x \to \infty$ by the theorem. See Geluk [G1], [G2] and [BGT, §6.1]. We note that we cannot apply [BI, Th. 4.1] to this kernel since this would require a zero-free region of $\zeta(z)$ of the form $1 - \epsilon < \Re z$ for some $\epsilon > 0$. This goes far beyond what is known; for the links between such zero-free regions and the error term in PNT, see [I, Th. 12.3].

§5. Double sums: $\rho = 0$

We turn now to the behaviour of double sums in the boundary case $\rho = 0$, which proves more delicate than the case $\rho > 0$ of §3.

We begin with the Abelian result:

THEOREM 5.1. *For $\ell \in R_0, \tilde{\ell}(x) := \ell(x)/\log x$, $f : [2, \infty) \to [0, \infty)$, (1.20) implies (1.21).*

We first prove the following extension of Pólya's theorem [P] (see also [PS, Part II, Ch. 4, No. 156]).

LEMMA 5.2. *With ℓ, f as above and g Riemann integrable on $[0, 1]$, (1.20) implies*

$$\frac{\log x}{x\ell(x)} \sum\nolimits_{p \le x} g(p/x) f(p) \to \int_0^1 g(t)dt. \tag{5.1}$$

In particular,

$$\frac{\log x}{x\ell(x)} \sum_{p \le x} f(p)\Big(\frac{x}{p} - [\frac{x}{p}]\Big) \to 1 - \gamma, \tag{5.2}$$

where γ is Euler's constant.

Proof. Let $a \in [0,1]$. By Theorem 2.0(i),

$$\frac{\log x}{x\ell(x)} \sum\nolimits_{p \le x} I_{[0,a]}(p/x) f(p) = \frac{\log x}{x\ell(x)} \sum\nolimits_{p \le ax} f(p) \to a. \tag{5.3}$$

Since g is Riemann integrable, for $\epsilon > 0$ we can approximate it above by H and below by h, where H, h are linear combinations of indicators $I_{[0,a]}$ $(a \in [0,1])$ with

$$\int_0^1 H(t)dt - \int_0^1 h(t)dt < \epsilon.$$

Then the sum $\sum_{p \le x} f(p) g(p/x)$ is approximated above and below by the same sums with H, h for g. For h,

$$\frac{\log x}{x\ell(x)} \sum_{p \le x} f(p) h(p/x) \to \int_0^1 h(t)dt,$$

by (5.3) and linearity, and similarly for H. Taking upper and lower limits, since $\epsilon > 0$ is arbitrary we obtain (5.1).

For (5.2), take

$$g(t) := \frac{1}{t} - [\frac{1}{t}] \qquad (0 < t \le 1)$$

and use

$$\int_0^1 \Big(\frac{1}{t} - [\frac{1}{t}]\Big) dt = 1 - \gamma$$

(Pólya [P]; [BGT, p. 296]).

Proof of Theorem 5.1. By (1.8)

$$\frac{1}{\lambda x} \sum_{n \le \lambda x} \sum_{p|n} f(p) - \frac{1}{x} \sum_{n \le x} \sum_{p|n} f(p) = \frac{1}{\lambda x} \sum_{p \le \lambda x} f(p)[\frac{\lambda x}{p}] - \frac{1}{x} \sum_{p \le x} f(p)[\frac{x}{p}]$$

$$= \frac{1}{\lambda x} \sum_{p \le \lambda x} f(p)\Big([\frac{\lambda x}{p}] - \frac{\lambda x}{p}\Big) + \sum_{x < p \le \lambda x} f(p)/p - \frac{1}{x} \sum_{p \le x} f(p)\Big([\frac{x}{p}] - \frac{x}{p}\Big).$$

Multiply through by $\log x/\ell(x)$ and let $x \to \infty$. By Lemma 5.2, the first and third terms on the right tend to $\gamma - 1$. By Theorem 2.1, the second term tends to $\log \lambda$. This gives (1.21) as required.

We turn now to the Tauberian converse, whose Tauberian content is Theorem 4.2.

THEOREM 5.3. *If $f : [2, \infty) \to [0, \infty)$ is non-decreasing and ℓ satisfies (1.22) and (1.23), then (1.21) implies (1.20).*

Proof. Step 1. Since $[2x/p] \geq 2[x/p]$,

$$\begin{aligned}
&\frac{1}{2x}\sum\nolimits_{p\leq 2x} f(p)[\frac{2x}{p}] - \frac{1}{x}\sum\nolimits_{p\leq x} f(p)[\frac{x}{p}] \\
&= \sum\nolimits_{p\leq x} f(p)\Big(\frac{1}{2x}[\frac{2x}{p}] - \frac{1}{x}[\frac{x}{p}]\Big) + \frac{1}{2x}\sum_{x<p\leq 2x} f(p)[\frac{2x}{p}] \\
&\geq \frac{1}{2x}\sum_{x<p\leq 2x} f(p)[\frac{2x}{p}] \geq \frac{f(x)}{2x}\sum_{x<p\leq 2x} 1 \\
&= \frac{f(x)}{2x}(\pi(2x) - \pi(x)) \sim \frac{f(x)}{2x}.\frac{x}{\log x},
\end{aligned}$$

by PNT. Using (1.8) and (1.21), the left $\sim (\log 2).\tilde{\ell}(x) = (\log 2).\ell(x)/\log x$. So the estimates above give

$$f(x) = O(\ell(x)). \tag{5.4}$$

Step 2. As before, we may take f continuous. By (1.8), (3.4) and integration by parts,

$$\sum_{n\leq x}\sum_{p|n} f(p) = \int_2^x \frac{f(t)}{\log t}[\frac{x}{t}]dt - \int_2^x R(t)[\frac{x}{t}]df(t) - \int_{[2,x]} R(t)f(t)d[\frac{x}{t}].$$

So for $x \geq 2$ and $\lambda > 1$, the difference

$$\frac{1}{\lambda x}\sum_{n\leq \lambda x}\sum_{p|n} f(p) - \frac{1}{x}\sum_{n\leq x}\sum_{p|n} f(p)$$

can be written as

$$I(\lambda x) - I(x) + II(\lambda x) - II(x) - III(\lambda x) + III(x),$$

where:

$$I(x) := \frac{1}{x}\int_2^x \frac{f(t)}{\log t}[\frac{x}{t}]dt, \qquad II(x) := \frac{1}{x}\int_{[2,x]} R(t)[\frac{x}{t}]df(t),$$

$$III(x) := \frac{1}{x}\int_{[2,x]} R(t)f(t)d[\frac{x}{t}].$$

We shall show

$$II(\lambda x) - II(x) = o(\tilde{\ell}(x)), \tag{5.5}$$

$$III(x) \to -\int_2^\infty \frac{f(u)R(u)}{u^2}. \tag{5.6}$$

From (5.6), $III(\lambda x) - III(x) = o(1)$. This is $o(\tilde{\ell}(x)) = o(\ell(x)/\log x)$, the accuracy we work to in the other terms, if the (restrictive) condition (1.23) holds.

Assuming (5.5), (5.6) for the moment, our assumption (1.21) now gives

$$I(\lambda x) - I(x) \sim \tilde{\ell}(x) \log \lambda.$$

This is (1.24), using Mellin-convolution notation, with k the Pólya kernel.

By (1.10), the Mellin transform $\check{k}(s) = \zeta(1+s)/(1+s)$ is holomorphic in $\Re s > -1$ except for a simple pole at the origin with residue 1, and has no zeros on the line $\Re s = 0$ ([I, Th. 1.5], or [T, II.3 Th. 9]). Since

$$[nt] \geq [n[t]] = n[t] \qquad (n \in \mathbb{N}, \quad t > 0),$$

we see that $k(nt) - k(t) \geq 0$ for $0 < t < \infty$. So k satisfies (k1) in §4 with $\rho = 0$ and $\mu_1 = 2$, $\mu_2 = 3$ say.

Since f is non-decreasing, we have for $\lambda > 1$

$$\begin{aligned} \inf_{t \in [1,\lambda]} \frac{\tilde{f}(tx) - \tilde{f}(x)}{\tilde{\ell}(x)} &= \inf_{t \in [1,\lambda]} \Big(\frac{f(tx)}{\log(tx)} - \frac{f(x)}{\log x} \Big) . \frac{\log x}{\ell(x)} \\ &\geq \frac{f(x)}{\ell(x)} . \frac{(-\log \lambda)}{\log(\lambda x)}, \end{aligned}$$

so by (5.4)

$$\liminf_{x \to \infty} \inf_{t \in [1,\lambda]} \frac{\tilde{f}(tx) - \tilde{f}(x)}{\tilde{\ell}(x)} \geq 0.$$

This gives the Tauberian condition (T) of Theorem 4.2 for $\tilde{f}$, $\tilde{\ell}$ and $\rho = 0$.

By Theorem 4.2, we now conclude $\tilde{f}(x) \sim \tilde{\ell}(x)$, or $f(x) \sim \ell(x)$, which is (1.20) as required.

Step 3. It remains to prove (5.5) and (5.6); here we prove (5.5). Now

$$II(\lambda x) - II(x) = \frac{1}{\lambda x} \int_x^{\lambda x} R(t) [\frac{\lambda x}{t}] df(t) + \int_2^x R(t) \Big(\frac{1}{\lambda x} [\frac{\lambda x}{t}] - \frac{1}{x} [\frac{x}{t}] \Big) df(t). \tag{5.7}$$

By the bounds (1.16) on R and (5.4) on f,

$$\begin{aligned} \frac{1}{\lambda x} \int_x^{\lambda x} R(t) [\frac{\lambda x}{t}] df(t) &<< \int_x^{\lambda x} e^{-\sqrt{\log t}} df(t) \\ &= o(\tilde{\ell}(x)) + \frac{1}{2} \int_x^{\lambda x} \frac{e^{-\sqrt{\log t}}}{t \sqrt{\log t}} . f(t) dt. \end{aligned}$$

Since $e^{-\sqrt{\log x}} \in R_0$, the integral on the right is, by (5.4), of order

$$\int_x^{\lambda x} \frac{e^{-\sqrt{\log t}}}{t\sqrt{\log t}}.\ell(t)dt \sim \frac{e^{-\sqrt{\log x}}}{\sqrt{\log x}}\ell(x). \int_1^{\lambda} du/u = o(\ell(x)/\log x) = o(\tilde{\ell}(x)).$$

So the first term on the right of (5.7) is negligible to the required accuracy. For the second, since

$$[\frac{\lambda x}{t}] + 1 \geq \frac{\lambda x}{t} \geq \lambda[\frac{x}{t}] \geq [\frac{\lambda x}{t}] - \lambda,$$

we have

$$|\frac{1}{\lambda x}[\frac{\lambda x}{t}] - \frac{1}{x}[\frac{x}{t}]| \leq \frac{1}{x},$$

hence

$$\int_2^x R(t)\Big(\frac{1}{\lambda x}[\frac{\lambda x}{t}] - \frac{1}{x}[\frac{x}{t}]\Big)df(t) << \frac{1}{x}\int_2^x te^{-\sqrt{\log t}}df(t).$$

Integrate by parts: the right is

$$O(x^{-1}) + e^{-\sqrt{\log x}}f(x) + \frac{1}{x}\int_2^x \Big(\frac{1}{2\sqrt{\log t}} - 1\Big)e^{-\sqrt{\log t}}f(t)dt.$$

By (5.4), this is of order

$$o(\tilde{\ell}(x)) + \frac{1}{x}\int_2^x e^{-\sqrt{\log t}}\ell(t)dt.$$

Since $\int_2^x e^{-\sqrt{\log t}}\ell(t)dt \sim x\ell(x)e^{-\sqrt{\log x}} = o(x\ell(x)/\log x) = o(x\tilde{\ell}(x))$, we see that the right hand side of (5.7) is $o(\tilde{\ell}(x))$, and this proves (5.5).

Step 4. It remains to prove (5.6); it is here that we will need the restriction (1.22) on ℓ (we have already used the more restrictive (1.23) in applying (5.6)). Set

$$g(x) := f(2/x)R(2/x) \qquad (0 < x \leq 1).$$

Then

$$\begin{aligned} 2x.III(2x) &= -\int_{(1,x]} R(2x/u)f(2x/u)d[u] \\ &= -\sum_{1<n\leq x} R(2x/n)f(2x/n) \\ &= -\sum_{1<n\leq x} g(n/x). \end{aligned}$$

Now by (1.16) and (5.4),

$$\int_0^1 |g(t)|dt = 2\int_2^{\infty} \frac{f(u)|R(u)|}{u^2}du << \int_2^{\infty} \ell(u)e^{-\sqrt{\log u}}du/u < \infty,$$

by (1.22). So $g \in L^1[0,1]$: g is Lebesgue integrable. To prove (5.6), it suffices to prove

$$\frac{1}{x}\sum_{1<n\le x} g(n/x) \to \int_0^1 g(t)dt \qquad (x\to\infty). \tag{5.8}$$

Now for any $\epsilon \in (0,1)$, g is bounded on $[\epsilon,1]$ (as f, R are locally bounded), and has at most countably many discontinuity points on $[\epsilon,1]$ (f is monotone, and R differs from the step-function π by the continuous function $\int_2^x dt/t\log t$). So g is Riemann integrable on $[\epsilon,1]$. So

$$\frac{1}{x}\sum_{\epsilon x<n\le x} g(n/x) \to \int_\epsilon^1 g(t)dt.$$

On the other hand, since

$$\frac{1}{t-1} \le \frac{2}{t} \le \frac{2}{[t]} \qquad (t\ge 2),$$

we have for $x \ge 1/\epsilon$

$$\begin{aligned}
\frac{1}{x}\sum_{1<n\le \epsilon x} |g(n/x)| &= \frac{1}{x}\sum_{1<n\le[\epsilon x]} |f(2x/n)R(2x/n)| \\
&<< \frac{1}{x}\int_2^{[\epsilon x]+1} f(2x/[t]).\frac{2x}{[t]}.\exp\{-\sqrt{\log(2x/[t])}\}dt \\
&\le \frac{1}{x}\int_2^{[\epsilon x]+1} f(2x/(t-1)).\frac{2x}{t-1}.\exp\{-\sqrt{\log(x/(t-1))}\}dt \\
&= \frac{1}{x}\int_1^{\epsilon x} f(2x/t).\frac{2x}{t}.\exp\{-\sqrt{\log(x/t)}\}dt \\
&= 2\int_{2x/[\epsilon x]}^{2x} \frac{f(u)}{u}.\exp\{-\sqrt{\log(u/2)}\}du \\
&\le 2\int_{2/\epsilon}^{\infty} \frac{f(u)}{u}.\exp\{-\sqrt{\log(u/2)}\}du.
\end{aligned}$$

By (5.4) and (1.22), the right can be made arbitrarily small for $\epsilon > 0$ small enough, uniformly in $x \ge 1/\epsilon$. Thus (5.8) follows, completing the proof.

Note. 1. Regarding (1.22), this comes from (1.16), which is not the best error term known for PNT. We could weaken (1.22), at some extra cost in complexity, but this is hardly worthwhile as (1.23) is much more restrictive.
2. We note also that (1.22) is unnecessary under the Riemann Hypothesis, or specifically, that ζ is zero-free in $\sigma > 1-\delta$ for some $\delta > 0$. For this implies that the error term $R(x)$ is $O(x^{1-\delta'})$ for some $\delta' > 0$. If we could use this in Step 4 of the proof above, we could

dispense with (1.22) with much to spare.
3. The best hope of avoiding (1.23) would be to replace (5.6) by a difference statement like (5.5), namely $III(\lambda x) - III(x) = o(\tilde{\ell}(x))$. Such a statement would require cancellation in the argument above, and we did not succeed in constructing such a proof.
4. The Lebesgue-integrable function g is unbounded, since fR is (by (1.16) and (1.20)). The validity of (5.8) suggests that g is directly Riemann integrable (see e.g. [Fe, XI.1]). This in turn suggests an alternative route to (5.8).

Appendix.

We gather here what we need on integration by parts for Lebesgue-Stieltjes integrals. For proofs, variants and background, see e.g. [M, p. 114], [S, p. 204], [BGT, Appendix 6.2].

We write $BV_{loc}(\mathbb{R})$ for the class of all $f : \mathbb{R} \to \mathbb{R}$ that are locally of bounded variation. Note that f need not be right-continuous here. Recall that, for $f \in BV_{loc}(\mathbb{R})$ the signed measure df on $\mathbb{R}$ is defined so that

$$df((c,d)) = f(d-) - f(c+) \qquad (-\infty < c < d < \infty),$$

whence $df(\{c\}) = f(c+) - f(c-)$ and

$$df((c,d]) = f(d+) - f(c+), \quad df([c,d)) = f(d-) - f(c-), \quad df([c,d]) = f(d+) - f(c-).$$

THEOREM A. *For $f, g \in BV_{loc}(\mathbb{R})$ with f right-continuous and g left-continuous, $-\infty < c < d < \infty$,*

$$\int_{(c,d]} g(t)df(t) = f(d)g(d) - f(c)g(c) - \int_{[c,d)} f(t)dg(t), \tag{A1}$$

$$\int_{[c,d]} g(t)df(t) = f(d)g(d+) - f(c-)g(c) - \int_{[c,d]} f(t)dg(t). \tag{A2}$$

THEOREM B. *With f, g as in Theorem A1 and $h \in BV_{loc}(\mathbb{R})$ continuous,*

$$\int_{[c,d]} f(t)d\{g(t)h(t)\} = \int_c^d f(t)g(t)dh(t) + \int_{[c,d]} f(t)h(t)dg(t).$$

REFERENCES

[B] N. H. Bingham, Tauberian theorems and the central limit theorem, *Annals of Probability* **9** (1981), 221-231.

[BGT] N. H. Bingham, C. M. Goldie and J. L. Teugels, "Regular variation", 2nd ed., Cambridge University Press, Cambridge, 1989 (1st ed. 1987).

[BI1] N. H. Bingham and A. Inoue, The Drasin-Shea-Jordan theorem for Fourier and Hankel transforms, *Quart. J. Math.* (2) **48** (1997), 279-307.

[BI2] N. H. Bingham and A. Inoue, An Abel-Tauber theorem for Hankel transforms, *Trends in Probability and Related Analysis* (Proc. Symp. Anal. Probab., Nat. Taiwan Univ., 1996, ed. N. Kono and N.-R. Shieh), 83-90, World Scientific, Singapore, 1997.

[BI3] N. H. Bingham and A. Inoue, Ratio Mercerian theorems with applications to Hankel and Fourier transforms, *Proc. London Math. Soc.* (3) **79** (1999), 626-648.

[BI4] N. H. Bingham and A. Inoue, Extension of the Drasin-Shea-Jordan theorem, *J. Math. Soc. Japan* **52**, No. 3 (2000), 15p.

[BI5] N. H. Bingham and A. Inoue, Tauberian and Mercerian theorems for systems of kernels, *J. Math. Anal. Appl.*, submitted.

[DeKI1] J.-M. De Koninck and A. Ivić, On the average prime factor of an integer and some related problems, *Ricerche di Matematica* **39** (1990), 131-140.

[DeKI2] J.-M. De Koninck and A. Ivić, Random sums related to prime divisors of an integer, *Publ. Inst. Mat. Beograd* **48** (62) (1990), 7-24.

[DeKI3] J.-M. De Koninck and A. Ivić, Arithmetic characterization of regularly varying functions, *Ricerche di Matematica* **44** (1995), 41-64.

[DeKKP] J.-M. De Koninck, I. Katai and B. M. Phong, A new characterization of the identity function, *J. Number Theory* **63** (1997), 325-338.

[DS] P. Diaconis and C. Stein, Some Tauberian theorems related to coin-tossing, *Annals of Probability* **6** (1978), 483-490.

[D] G. Doetsch, "Theorie und Anwendung der Laplace-Transformation", Springer-Verlag, Berlin, 1937 (Grundl. Math. Wiss. 47).

[Fe] W. Feller, "An introduction to probability theory and its applications", Volume 2, 2nd ed., Wiley, New York, 1971 (1st ed. 1966).

[Fo] G. B. Folland, "A course in abstract harmonic analysis", CRC Press, Boca Raton, 1995.

[G1] J. L. Geluk, An Abel-Tauber theorem on convolutions with the Möbius function, *Proc. Amer. Math. Soc.* **77** (1979), 201-210.

[G2] J. L. Geluk, An Abel-Tauber theorem for partitions, *Proc. Amer. Math. Soc.* **82** (1981), 571-575.

[H] G. H. Hardy, "Divergent series", Oxford Univ. Press, Oxford, 1949.

[HR] G. H. Hardy and S. Ramanujan, Asymptotic formulae for the distribution of integers of various types, *Proc. London Math. Soc.* (2) **16** (1917), 112-132 (reprinted in "Collected

Works of G. H. Hardy", Volume II, Part 2(a), Oxford Univ. Press, Oxford).
[HW] G. H. Hardy and E. M. Wright, "An introduction to the theory of numbers", 4th ed., Oxford Univ. Press, Oxford, 1960.
[I] A. Ivić, "The Riemann zeta function", Wiley, New York, 1985.
[M] P.-A. Meyer, "Probability and potentials", Blaisdell, 1966.
[P] G. Pólya, Über eine neue Weise, bestimmte Integrale in der analytischen Zahlentheorie zu gebrauchen, *Göttinger Nachrichten* (1917), 149-159.
[PS] G. Pólya and G. Szegö, *Problems and theorems in analysis I*, Classics in Mathematics, Springer-Verlag, New York, 1998 (reprint of the 1978 edition).
[S] A. N. Shiryaev, *Probability*, Springer-Verlag, Heidelberg, 1984.
[T] G. Tenenbaum, "Introduction to analytic and probabilistic number theory" (Cambridge Studies in Advanced Math. **46**), Cambridge Univ. Press, Cambridge, 1995 (translation of the French edition, Univ. Nancy, 1990).
[Ti] E. C. Titchmarsh, "The theory of the Riemann zeta function", 2nd ed. (revised by D. R. Heath-Brown), Oxford Univ. Press, Oxford, 1986.
[W] D. V. Widder, "The Laplace transform", Princeton Univ. Press, Princeton, NJ, 1941.