MODELLING ASSET RETURNS WITH HYPERBOLIC DISTRIBUTIONS [1]

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ABSTRACT In this note we discuss applications of the hyperbolic distributions in financial modelling. In particular we discuss approaches to modelling stock returns and interest rates, using a modelling based on hyperbolic Lévy processes. We consider the structure of the hyperbolic model, its incompleteness, choice of equivalent martingale measure, option pricing and hedging, and Value at Risk. We also give some empirical studies fitting the model to real data. The moral of this survey is simply this: if one wants a model that goes beyond the benchmark Black-Scholes model, but not as far as the complications of, say, stochastic-volatility models, the hyperbolic model is a good candidate for the model of first choice.

1 Introduction

The benchmark theory of mathematical finance is the Black-Scholes theory, based on the Wiener process in the continuous-time setting or appropriate discrete-time versions such as binomial trees. This has the virtues of being mathematically tractable and well-known, but the equally well-known drawback of not corresponding to reality. Consequently, much work has been done on attempts to generalize the Wiener-based Black-Scholes theory to more complicated models chosen to provide a better fit to empirical data, preferably with a satisfactory theoretical basis also. We focus here on models including the hyperbolic distributions. This family has been used to model financial data by several authors, including Eberlein & Keller [16] and Bibby & Sørensen [7]; much of the underlying work derives from the Danish school of Barndorff-Nielsen and co-workers.

We mention briefly various other approaches to generalisations of the Wiener-based Black-Scholes theory. One of the more immediately apparent deficiencies of the Black-Scholes model is the tail behaviour: most financial data exhibit thicker tails than the faster-than-exponentially decreasing tails of the normal distribution. Replacement of the normal law by a stable distribution, whose tails decrease much

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more slowly – like a power $x^{-\alpha}$ – is an idea dating back to Mandelbrot's work in the 1960s; for a recent textbook synthesis of this line of work see Mandelbrot [35]. However, it is nowadays recognized that the tails of most financial time series have to be modeled with $\alpha > 2$ (see Pagan [37]), while stable distributions correspond to $\alpha \in (0,2)$.

In addition to this, stochastic volatility models and ARCH and GARCH models from time series have been used, e.g. by Hull and White in [27] and by Duan [14]; overviews are given in [21, 23, 26].

We turn in §2 below to a description of the hyperbolic distribution and theory used in modelling financial data. The principal complication is that hyperbolic-based models of financial markets are incomplete (stochastic volatility models share this drawback; for a recent alternative approach see Rogers [39]). Consequently, equivalent martingale measures are no longer unique, and we thus face the question of choosing an appropriate equivalent martingale measure for pricing purposes. We discuss the relevant theory in §3. We discuss option pricing, hedging and Value at Risk (VaR) in the framework of a case study in §4.

2 Hyperbolic models of financial markets and hyperbolic Lévy motion

We begin with the basic stochastic differential equation (SDE) of Black-Scholes theory for the price process $S = (S_t)$,

$$dS_t = S_t(\mu dt + \sigma dW_t), \tag{1}$$

where μ is the drift (mean growth rate), σ the volatility, and $W = (W_t)$ – the driving noise process – a Wiener process or Brownian motion. The solution of the SDE (1) is

$$S_t = S_0 \exp\{(\mu - \sigma^2/2)t + \sigma W_t\}, \tag{2}$$

the stochastic exponential of the drifting Brownian motion $\mu dt + \sigma dW_t$. For proof and references see e.g. [9], §5.6.1.

Now the driving noise process W is a Lévy process – a stochastic process with stationary independent increments (for a monograph treatment of Lévy processes see Bertoin [6]). Stationarity is a sensible assumption – at least for modelling markets in equilibrium on not too large a timescale – and although the independent increments assumption is certainly open to question, it is reasonable to a first approximation, which is all we attempt here. What singles out the Wiener process W among Lévy processes is path-continuity. Now the driving noise represents the net effect of the random buffeting of the multiplicity of factors at work in the economic environment, and one would expect that, analysed closely, this would be discontinuous, as the individual 'shocks' – pieces of price-sensitive information – arrive. (Indeed, price processes themselves are discontinuous looked at closely enough: in addition to the discrete shocks, one has discreteness of monetary values and the effect on supply and demand of individual transactions.)

One is thus led to consider a SDE for the price process $Y = (Y_t)$ of the form

$$dY_t = bY_{t-}dt + \sigma Y_{t-}dZ_t, \tag{3}$$

with $Z = (Z_t)$ a suitable driving Lévy process. Now a Lévy process, or its law, is characterised via the Lévy-Khintchine formula by a drift a, the variance σ of any Gaussian (Wiener, Brownian) component, and a jump measure $d\mu$. Since the form of $d\mu$ is constrained only by integrability restrictions, such a model would be non-parametric. While the modelling flexibility of such an approach, coupled with the theoretical power of modern non-parametric statistics, raises interesting possibilities, these would take us far beyond our modest scope here. We are led to seek suitable parametric families of Lévy processes, flexible enough to provide realistic models and tractable enough to allow empirical estimation of parameters from actual financial data. We refer to Chan [13] for a thorough theoretical analysis of models of price processes with driving noise a general Lévy process.

One such family has been mentioned in §1: the stable process. There are four parameters, corresponding to location and scale (the two 'type' parameters one must expect in a statistical model), plus two 'shape' parameters, α (governing tail decay: $0 < \alpha < 2$, with $\alpha = 2$ giving Brownian motion) and β, a skewness or asymmetry parameter. Our concern here is the hyperbolic family, again a four-parameter family with two type and two shape parameters. Recall that, for normal (Gaussian) distributions, the log-density is quadratic – that is, parabolic – and the tails are very thin. The hyperbolic family is specified by taking the log-density instead to be hyperbolic, and this leads to thicker tails as desired (but not as thick as for the stable family).

Before turning to the specifics of notation, parametrisation, etc., we comment briefly on the origin and scope of the hyperbolic distributions. Both the definition and the bulk of applications stem from Barndorff-Nielsen and co-workers. Thus [3] contains the definition and an application to the distribution function of particle size in a medium such as sand (see also [4]). Later, in [4], hyperbolic distribution functions are used to model turbulence. Now the phenomenon of atmospheric turbulence may be regarded as a mechanism whereby energy, when present in localised excess on one volume scale in air, cascades downwards to smaller and smaller scales (note the analogy to the decay of larger particles into smaller and smaller ones in the sand studies). Barndorff-Nielsen had the acute insight that this 'energy cascade effect' might be paralleled in the 'information cascade effect', whereby price-sensitive information originates in, say, a global newsflash, and trickles down through national and local level to smaller and smaller units of the economic and social environment. This insight is acknowledged by Eberlein and Keller [16] (see also [17, 18]), who introduced hyperbolic distribution functions into finance and gave detailed empirical studies of its use to model financial data, particularly daily stock returns. Further and related studies are [7, 13, 15, 34, 41, 43].

To return to the Lévy process, recall (see e.g. [6]) that the sample path of a Lévy process $Z = (Z_t)$ can be decomposed into a drift term bt, a Gaussian or Wiener term, and a pure jump function. This jump component has finitely or infinitely many jumps in each time-interval, almost surely, according to whether the Lévy or jump measure is finite or infinite. Of course, the latter case is unrealistic in detail – but so are all models. It is, however, better adapted to modelling most financial data than the former. There, the influence of individual jumps is visible, indeed predominates, and we are in effect modelling shocks. This is appropriate for phenomena such as stock market crashes, or markets dominated by 'big players', where individual trades shift prices. To model the everyday movement of ordinary quoted stocks under the market pressure of many agents, an infinite measure is appropriate. Incidentally, a penetrating study of the mechanism whereby the actions of economic agents are translated into market forces and price movements has recently been given by Peskir and Shorish [38].

We need some background on Bessel functions (see [45]). Recall the Bessel functions J_ν of the first kind ([45], §3.11), Y_ν of the second kind ([45], §3.53), and K_ν ([45], §3.7), there called a Bessel function with imaginary argument or Macdonald function, nowadays usually called a Bessel function of the third kind. From the integral representation

$$K_\nu(x) = \frac{1}{2}\int_0^\infty u^{\nu-1}\exp\left\{-\frac{1}{2}x(u+1/u)\right\}du \quad (x>0) \tag{4}$$

([45],§6.23) one sees that

$$f(x) = \frac{(\psi/\chi)^{\frac{1}{2}\lambda}}{2K_\lambda(\sqrt{\chi\psi})}x^{\lambda-1}\exp\left\{-\frac{1}{2}(\psi x+\chi/x)\right\} \quad (x>0) \tag{5}$$

is a probability density function. The corresponding law is called the generalized inverse Gaussian $GIG_{\lambda,\psi,\chi}$; the inverse Gaussian is the case $\lambda = 1$: $IG_{\chi,\psi} = GIG_{1,\psi,\chi}$. These laws were introduced by Good in 1953 (see [24]); for a monograph treatment of their statistical properties, see Jørgensen [29], and for their role in models of financial markets, [44], III, 1.d.

Now consider a Gaussian (normal) law $N(\mu + \beta\sigma^2, \sigma^2)$ where the parameter σ^2 is random and is sampled from $GIG_{1,\psi,\chi}$. The resulting law is a mean-variance mixture of normal laws, the mixing law being generalised inverse Gaussian. It is written $\mathbb{E}_{\sigma^2} N(\mu + \beta\sigma^2, \sigma^2)$; it has a density of the form

$$\frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1(\delta\sqrt{\alpha^2 - \beta^2})} \exp\left\{-\alpha\sqrt{\delta^2 + (x-\mu)^2} + \beta(x-\mu)\right\} \tag{6}$$

([3]), where $\alpha^2 = \psi + \beta^2$ and $\delta^2 = \chi$. Just as the Gaussian law has log-density a quadratic – or parabolic – function, so this law has log-density a hyperbolic function. It is accordingly called a hyperbolic distribution. Various parametrisations are possible. Here μ is a location and δ a scale parameter, while $\alpha > 0$ and β $(0 \le |\beta| < \alpha)$ are shape parameters. One may pass from (α, β) to (ϕ, γ) via

$$\alpha = (\phi + \gamma)/2,\ \beta = (\phi - \gamma)/2,\ \text{so } \phi\gamma = \alpha^2 - \beta^2,$$

and then to (ξ, χ) via

$$\xi = (1 + \delta\sqrt{\phi\gamma})^{-\frac{1}{2}},\ \chi = \frac{\xi\beta}{\alpha} = \xi\frac{\phi - \gamma}{\phi + \gamma}.$$

This parametrisation (in which ξ and χ correspond to the classical shape parameters of skewness and kurtosis) has the advantage of being affine invariant (invariant under changes of location and scale). The range of (ξ, χ) is the interior of a triangle

$$\nabla = \{(\xi, \chi) : 0 \le |\chi| < \xi < 1\},$$

called the shape traingle (see figure 1). It suffices for our purpose to restrict to the centred ($\mu = 0$) symmetric ($\beta = 0$, or $\chi = 0$) case, giving the two-parameter family of densities (writing $\zeta = \xi^{-2} - 1$)

$$hyp_{\zeta,\delta}(x) = \frac{1}{2\delta K_1(\zeta)} \exp\left\{-\zeta\sqrt{1 + \left(\frac{x}{\delta}\right)^2}\right\}, \quad (\zeta, \delta > 0). \tag{7}$$

Infinite divisibility. Recall (Feller [20], XIII,7, Theorem 1) that a function ω is the Laplace transform of an infinitely divisible probability law on $\mathbb{R}_+$ iff

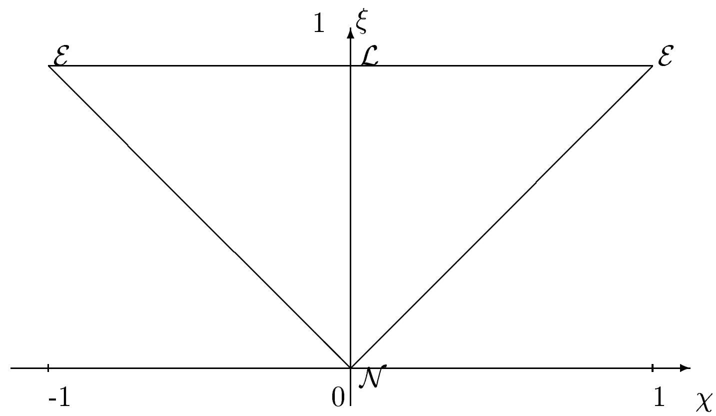

Figure 1: Shape Triangle

$\omega = e^{-\psi}$, where $\psi(0) = 0$ and ψ has a completely monotone derivative (that is, the derivatives of ψ' alternate in sign). Grosswald ([25]) showed that if

$$Q_\nu(x) := K_{\nu-1}(\sqrt{x})/(\sqrt{x}K_\nu(\sqrt{x})) \quad (\nu \geq 0, x > 0),$$

then Q_ν is completely monotone. Hence Barndorff-Nielsen and Halgreen ([5]) showed that the generalised inverse Gaussian laws GIG are infinitely divisible. Now the GIG are the mixing laws giving rise to the hyperbolic laws as normal mean-variance mixtures. This transfers infinite divisibility (see e.g. Kelker [32], Keilson and Steutel [31], §§1,2), so the hyperbolic laws are infinite divisible.

Characteristic functions. The mixture representation transfers to characteristic functions on taking the Fourier transform. It gives the characteristic function of $hyp_{\zeta,\delta}$ as

$$\phi(u) = \phi(u;\zeta,\delta) = \frac{\zeta}{K_1(\zeta)} \frac{K_1\left(\sqrt{\zeta^2+\delta^2u^2}\right)}{\sqrt{\zeta^2+\delta^2u^2}}. \tag{8}$$

If $\phi(u)$ is the characteristic function of Z_1 in the corresponding Lévy process $Z = (Z_t)$, that of Z_t is $\phi_t = \phi^t$. The mixture representation of $hyp_{\zeta,\delta}$ gives

$$\phi_t(u) = \exp\{tk(\frac{1}{2}u^2)\},$$

where $k(.)$ is the cumulant generating function of the law IG,

$$I\!E\left(e^{-sY}\right) = e^{k(s)},$$

where Y has law $IG_{\psi,\chi}$ (recall $\chi = \delta^2$), and Grosswald's result above is

$$Q_\nu(t) = \int_0^\infty q_\nu(x)dx/(x+t),$$

where

$$q_\nu = 2/(\pi^2 x(J_\nu^2(\sqrt{x}) + Y_\nu(\sqrt{x}))^2) > 0 \quad (x > 0)$$

(thus Q_ν is a Stieltjes transform, or iterated Laplace transform, [46], VIII). Using this and the Lévy-Khintchine formula Eberlein and Keller [16] obtained the density $\nu(x)$ of the Lévy measure $\mu(dx)$ of Z as

$$\nu(x) = \frac{1}{\pi^2|x|}\int_0^\infty \frac{\exp\left\{-|x|\sqrt{2y+(\zeta/\delta)^2}\right\}}{y\left(J_1^2(\delta\sqrt{2y}) + Y_1^2(\delta\sqrt{2y})\right)}dy + \frac{\exp\{-|x|\zeta/\delta\}}{|x|} \tag{9}$$

and then

$$\phi_t(u) = \exp\{tK(\frac{1}{2}u^2)\}, \quad K(\frac{1}{2}u^2) = \int_{-\infty}^\infty \left(e^{iux} - 1 - iux\right)\nu(x)dx.$$

Now [45], §7.21

$$\left.\begin{array}{rcl} J_\nu(x) & \sim & \sqrt{2/\pi x}\cos\left(x - \frac{1}{2}\pi\nu - \frac{1}{4}\pi\right), \\ Y_\nu(x) & \sim & \sqrt{2/\pi x}\sin\left(x - \frac{1}{2}\pi\nu - \frac{1}{4}\pi\right), \end{array}\right. \quad (x \to \infty).$$

So the denominator in the integral in (9) is asymptotic to a multiple of $y^{\frac{1}{2}}$ as $y \to \infty$. The asymptotics of the integral as $x \downarrow 0$ are determined by that of the integral as $y \to \infty$, and (writing $\sqrt{2y+(\zeta/\delta)^2}$ as t, say) this can be read off from the Hardy-Littlewood-Karamata theorem for Laplace transforms (Feller [20], XIII.5, Theorem 2, or Bingham-Goldie-Teugels, [8], Theorem 1.7.1). We see that $\nu(x) \sim c/x^2$, $(x \downarrow 0)$ for c a constant. In particular the Lévy measure is infinite, as required.

From driving noise to asset returns. Returning to the SDE (3), with driving noise a hyperbolic Lévy process Z as above, the solution is given by the stochastic exponential

$$Y(t) = Y(0)\exp\left\{Z^{\zeta,\delta}(t) + \rho t\right\}\prod_{0<s\le t}\left(1 + \Delta Z^{\zeta,\delta}(s)\right)e^{-\Delta Z^{\zeta,\delta}(s)} \tag{10}$$

(here the quadratic variation $[Z]_t$ is just

$$\sum_{s \leq t} (\Delta Z_s)^2,$$

with no continuous component, as Z is a pure jump process). Passing to logarithms to pass from prices to returns, one obtains two terms, the hyperbolic term $Z_t + \rho t$ and the sum-of-jumps term. To first order, this is $\sum_{s \leq t} (\Delta Z_s)^2$. Now since the Lévy measure is infinite, small jumps predominate, and these become second order effects when squared, so negligible. Thus to a first approximation, the return process is hyperbolic.

Tails and Shape. The classic empirical studies of Bagnold [1, 2] reveal the characteristic pattern that, when log-density is plotted against log-size of particle, one obtains a unimodal curve approaching linear asymptotics at $\pm\infty$. Now the simplest such curve is the hyperbola, which contains four parameters: location of the mode, the slopes of the asymptotics, and curvature near the mode (the modal height is absorbed by the density normalisation). This is the empirical basis for the hyperbolic laws in particle-size studies. Following Barndorff-Nielsen's suggested analogy, a similar pattern was sought, and found, in financial data, with log-density plotted against log-price. Studies by Eberlein and co-workers [16, 17, 18], Bibby and Sørensen [7], Rydberg [42, 43] and other authors show that hyperbolic densities provide a good fit for a range of financial data, not only in the tails but throughout the distribution. The hyperbolic tails are log-linear: much fatter than normal tails but much thinner than stable ones.

Hyperbolic diffusion model. We pointed out that the weakness of the hyperbolic Lévy process model lies in the independent-increments assumption. This is avoided in the hyperbolic diffusion model of Bibby and Sørensen [7]. They use a stochastic volatility $v(X_s)$, where $dX_t = v(X_t) dW_t$. For $v^2(.)$ log-hyperbolic, this gives rise to an ergodic diffusion, whose invariant distribution is hyperbolic. See Bibby and Sørensen [7] §2 for the model, §3 for its fit to real financial data and §4 for option pricing.

3 Equivalent martingale measure

As in the other non-normality approaches mentioned above, the drawback of the model is that the underlying stochastic model of the financial market becomes incomplete. We thus face the question of choosing an appropriate equivalent martingale measure for pricing purposes. We outline here two approaches to determining an equivalent martingale measure, the risk-neutral Esscher measure and the minimal martingale measure.

3.1 General Lévy-process based financial market model

Recall our Lévy process based model of a financial price process:

$$dY_t = bY_{t-}dt + \sigma Y_{t-}dZ_t, \tag{3}$$

with $Z = (Z_t)$ a suitable driving Lévy process on a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. The characteristic function takes the form

$$\mathbb{E}\left(\exp\{i\theta Z_t\}\right) = \exp\{-t\psi(\theta)\}$$

with ψ the Lévy exponent of Z. The Lévy-Khintchine formula implies

$$\begin{aligned}\psi(\theta) \quad = \quad & \frac{c^2}{2}\theta^2 + i\alpha\theta + \int\limits_{\{|x|<1\}} \left(1 - e^{-i\theta x} - i\theta x\right)\mu(dx) \\ & + \int\limits_{\{|x|\geq 1\}} \left(1 - e^{-i\theta x}\right)\mu(dx)\end{aligned}$$

with $\alpha, c \in \mathbb{R}$ and μ a σ-finite measure on $\mathbb{R}/\{0\}$ satisfying

$$\int \min\{1, x^2\}\mu(dx) < \infty.$$

μ is called the Lévy measure.

From the Lévy-Khintchine formula we deduce the Lévy decomposition of Z, which says that Z must be a linear combination of a standard Brownian motion W and a pure jump process X independent of W (a process is a pure jump process if its quadratic variation is simply $\langle X\rangle = \sum_{0<s\leq t}(\Delta X)^2$). We write

$$Z_t = cW_t + X_t. \tag{11}$$

Under further assumptions on Z_t we can find a Lévy decomposition of X (for details see [13], §2 or [44], III §1b and VII §3c). This leads to the decomposition

$$Z_t = cW_t + M_t + at, \tag{12}$$

where M_t is a martingale with $M_0 = 0$ and $a = \mathbb{E}(X_1)$. We shall assume the existence of such a decomposition (12). Then we can restate (3) as

$$dY_t = (a\sigma + b)Y_{t-}dt + \sigma Y_{t-}(cdW_t + dM_t), \tag{13}$$

where the coefficients b and σ are constants (though one can generalise to deterministic functions). Now (13) has an explicit solution

$$\begin{aligned} Y_t \quad = \quad & Y_0 \exp\left\{ \int_0^t c\sigma dW_s + \int_0^t \sigma dM s + \int_0^t \left(a\sigma + b - \frac{\sigma^2 c^2}{2} \right) ds \right\} \\ \times & \prod_{0<s\leq t} (1 + \sigma \Delta M_s) \exp\{-\sigma \Delta M_s\}. \end{aligned}$$

In order to ensure that $Y_t \geq 0$ for all t almost surely, we need $\sigma \Delta M_t \geq -1$ for all t. A sufficient condition is that the jumps of X should be suitably bounded from below.

We also introduce the (locally) risk-free bank account (short rate) process B_t with

$$dB_t = r_t B_t dt, \tag{14}$$

with r_t a suitable process.

3.2 Existence of equivalent martingale measures

To characterise equivalent martingale measures $\mathbb{Q}$ under which discounted price processes $\tilde{S}_t = S_t/B_t$ are (local) $\mathcal{F}_t$-martingales, we rely on Girsanov's theorem for semi-martingales. (See Jacod and Shiryaev [28], III §3d, for a thorough treatment, or [44], VII §3g for a textbook summary. Bühlmann et al. [11] provide a discussion geared towards financial applications.) We follow the exposition in [13], to which we refer for technical details. Define a process L_t as

$$L_t = 1 + \int_0^t G_s L_{s-} dB_s + \int_0^t \int_{\mathbb{R}} L_{s-}[H(s,x) - 1] M(ds, dx), \tag{15}$$

with functions G and H satisfying certain regularity conditions. Then

Theorem 3.1. *Assume* $\mathbb{Q}$ *is absolutely continuous with respect to* $\mathbb{P}$ *on* $\mathcal{F}_T$, *and*

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t$$

with $\mathbb{E}(L_T) = 1$. Under $\mathbb{Q}$ the process

$$\tilde{W}_t = W_t - \int_0^t G_s ds$$

is a standard Brownian motion and the process X is a quadratic pure jump process with compensator measure given by

$$\tilde{\nu}(dt, dx) = dt\tilde{\nu}_t(dx),$$

where

$$\tilde{\nu}_t(dx) = H(t,x)\nu(dx)$$

and the previsible part is given by

$$\tilde{a}_t = \mathbb{E}_{\mathbb{Q}}(X_t) = at + \int_0^t \int_I Rx(H(s,x) - 1)\nu(dx)ds.$$

Using Theorem 3.1 we can write the discounted process $\tilde{S}$ in terms of the $\mathbb{Q}$ martingale $\tilde{M}$ and the $\mathbb{Q}$ Brownian motion $\tilde{W}$ and read off a necessary and sufficient condition for $\tilde{S}$ to be a $\mathbb{Q}$ martingale:

$$c\sigma_t G_t + a\sigma_t + b_t - r_t + \int_I R\sigma_s x(H(s,x) - 1)\nu(dx) = 0. \tag{16}$$

Since the martingale condition (16) doesn't specify the functions G and H uniquely, we have an infinite number of equivalent martingale measures, i.e. the market model is incomplete. We hence face the problem of choosing a particular martingale measure for pricing (and hedging) contingent claims.

3.3 Choice of an equivalent martingale measure

We briefly discuss two widely used approaches (for an overview see Bingham and Kiesel [9], chapter 7).

Minimal martingale measure. Consider the problem of hedging a contingent claim H with maturity T (modelled as a bounded $\mathcal{F}_T$-measurable random variable) in an incomplete financial market model. Under an equivalent martingale measure $\mathbb{Q}$ we only can obtain a representation of the form

$$\tilde{H} = H_0 + \int_0^T \xi_t d\tilde{S}_t + L_T,$$

where (L_t) is a square-integrable martingale orthogonal to the martingale part of $\tilde{S}$ under $\mathbb{P}$. ξ corresponds to a trading strategy which would reduce the remaining risk to the intrinsic component of the contingent claim. Therefore we try to find a martingale measure that allows for such a decomposition and preserves orthogonality. Such a measure is called minimal martingale measure.

Esscher transforms. The idea here is to define equivalent measures via

$$\left.\frac{d\mathbb{P}_\theta}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left\{ -\int_0^t \theta_s Z_s ds + \int_0^t \psi(\theta_s) ds \right\}, \tag{17}$$

where $\psi(\theta) = -\log \mathbb{E}(\exp(-\theta Z_1))$ is the Lévy exponent of Z given by (3). One then has to choose θ_s to satisfy the martingale conditions. The use of Esscher transforms as a technical tool has a long history in actuarial sciences. Gerber and Shiu [22] were the first to introduce it systematically to option pricing. Chan [13] provides an interpretation of it in terms of entropy – the measure $\mathbb{P}$ encapsulates information about market behaviour, then pricing by Esscher transforms amounts to choosing the equivalent martingale measure which is closest to $\mathbb{P}$ in terms of information content. Equilibrium based justifications have been given in [12, 22]. Further background information can be found in [13], [9], §7.3 and [44], VII §3c.

We outline an approach suggested by Rogers [40] (for a general discussion of optimal consumption/investment problems see [30, 33]). Consider a financial market defined as in §3.1 with a discount process $\beta(t) = e^{-\delta t}$, $\delta > 0$ a constant and introduce a representative agent with a utility function U. Suppose that the wealth process of the investor satisfies

$$dX_t = rX_t dt + \pi_t \left(\frac{dY_t}{Y_{t-}} - r dt \right) - C_t dt, \tag{18}$$

with (π_t) resp. (C_t) the portfolio resp. consumption process of the investor. The return process dY_t/Y_{t-} is given as in (3) with a suitable driving Lévy process. The investor wishes to maximise

$$\mathbb{E}\left(\int_0^\infty \exp\{-\delta t\} U(C_t) dt \right). \tag{19}$$

We specialize to a utility function $U(x) = -\gamma^{-1}e^{-\gamma x}$ and solve the investors optimisation problem following the standard Hamilton-Jacobi-Bellman approach (see Karatzas and Shreve [30], §5.8, or Korn [33]). This leads to an optimal consumption process C_t^* (and an optimal portfolio process π^*).

Now the equivalent martingale measure is given by

$$e^{-rt} \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{Y}_t} = e^{-rt}L_t \propto e^{-\delta t}U'(C_t^*). \tag{20}$$

Solving (20) leads to

$$L_t = \exp\{-\theta^* Z_t + \psi(\theta^*)t\}$$

with an optimal parameter θ^* , which is exactly of form (17).

4 Case study

4.1 Fitting the hyperbolic distribution

It is well known that the normal distribution fits stock returns poorly. In this section we compare the normal fit with the fit obtained by using the hyperbolic distribution (similar studies are contained in Eberlein and Keller [16], Rydberg [43]). As an example we consider daily BMW returns during the period September 1992 – July 1996, i.e. a total of 1000 data points. We fit the normal distribution using the standard estimators for mean and variance. To estimate the parameters of the hyperbolic distribution we use a computer program described in Blaesild and Sørensen [10]. Under the assumptions of independence and identical distribution a maximum likelihood analysis is performed. The maximum likelihood estimates of the parameters are

$$\begin{array}{ll} \hat{\alpha} = 89.72 & \hat{\beta} = 4.7184 \\ \hat{\delta} = 0.0009 & \hat{\mu} = -0.0015 \end{array}$$

Figure 2 shows the corresponding empirical density, the normal density and the hyperbolic density. Figure 2 indicates there is more mass around the origin and in the tails than the normal distribution suggests and that fitting returns to a hyperbolic distribution is to be preferred. The same conclusion is made even more clearly in the wider range of empirical studies, and the accompanying density plots, given by Eberlein and Keller [16].

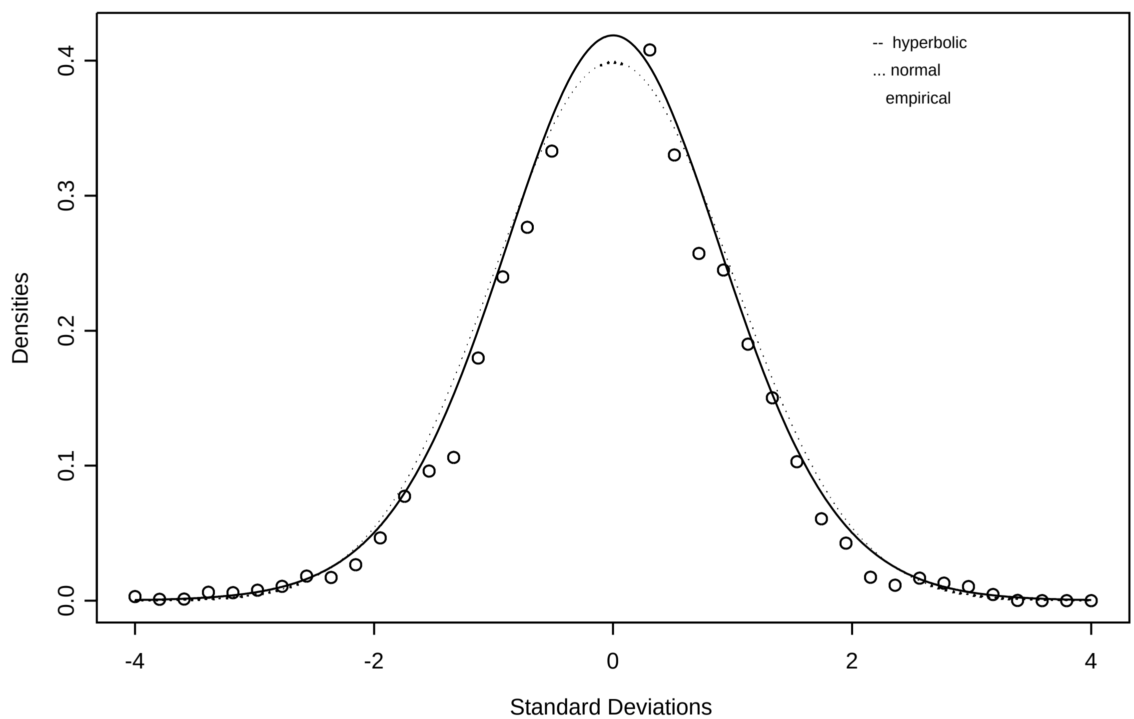

Figure 2: Density Plots

4.2 Constructing the hyperbolic Lévy motion

Given the empirical findings in §4.1 it is natural to concentrate now on the symmetric centered case, i.e. set $\mu = \beta = 0$. This leads to modelling the stock-price process by (10) (i.e. (3) with driving noise a hyperbolic Lévy process). As mentioned above the return process so generated is hyperbolic to a first approximation. To generate exactly hyperbolic returns along time-intervals of length 1 Eberlein and Keller [16] suggest writing

$$S(t) = S(0) \exp\{Z^{\zeta,\delta}(t)\} \tag{21}$$

as a model for stock prices, and we shall work with this model in the sequel.

4.3 The risk-neutral Esscher measure

To price contingent claims in the hyperbolic Lévy model we use Esscher transforms, which are defined via (17). Now in our model (21) the function $\theta(.)$ in (17) reduces to a constant. Therefore defining the moment-generating funcion of $Z^{\zeta,\delta}(t)$ as

$$M(\theta,t) = \mathbb{E}\left[e^{\theta Z^{\zeta,\delta}(t)}\right] \tag{22}$$

the Esscher transforms are defined by

$$L(t) = \left\{e^{\theta Z^{\zeta,\delta}(t)} M(\theta,1)^{-t}\right\}_{t\geq 0} \tag{23}$$

(observe that L is a positive martingale). According to (17) we define equivalent measures via

$$\left.\frac{d\mathbb{P}_\theta}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L(t)$$

and call $\mathbb{P}_\theta$ the Esscher measure of parameter θ.

The risk-neutral Esscher measure is the Esscher measure of parameter $\theta = \theta^*$ such that the process

$$\left\{e^{-rt}S(t)\right\}_{t\geq 0} \tag{24}$$

is a martingale (with r the daily interest rate). From the martingale condition

$$\mathbb{E}\left[e^{-rt}S(t);\theta^*\right] = S(0)$$

we find

$$e^r = \frac{M(1+\theta^*,1)}{M(\theta^*,1)},$$

from which the parameter θ^* is uniquely determined. Indeed, since the moment generating function $M^{\zeta,\delta}(u,1)$ is

$$M^{\zeta,\delta}(u,1) = \frac{\zeta}{K_1\zeta}\frac{K_1(\sqrt{\zeta^2-\delta^2u^2})}{\sqrt{\zeta^2-\delta^2u^2}}, \quad |u| < \frac{\zeta}{\delta},$$

we have

$$r = \log\left[\frac{K_1\left(\sqrt{\zeta^2-\delta^2(\theta+1)^2}\right)}{K_1\left(\sqrt{\zeta^2-\delta^2\theta^2}\right)}\right] - \frac{1}{2}\log\left[\frac{\zeta^2-\delta^2(\theta+1)^2}{\zeta^2-\delta^2\theta^2}\right]. \tag{25}$$

Given the daily interest rate r and the parameters ζ,δ equation (25) can be solved by numerical methods for the martingale parameter θ^*.

4.4 Option pricing

A useful tool for option pricing in the hyperbolic model (21) (and indeed any model of type $S(t) = S(0)\exp\{X(t)\}$ with $X(t)$ a process with independent and stationary increments) is the following (compare [22])

Lemma 4.1 (Factorisation formula). *Let g be a measurable function and h, k and t be real numbers, $\theta \geq 0$, then*

$$I\!\!E\left[S(t)^k g(S(t));\right] = I\!\!E\left[S(t)^k; h\right] I\!\!E\left[g(S(t)); k+h\right]. \tag{26}$$

We now value a European call with maturity T and strike K in the hyperbolic model, that is we assume that the underlying $S(t)$ has price dynamics given by (21). By the risk-neutral valuation principle we have to calculate

$$\begin{aligned} I\!\!E\left[e^{-rT}(S(T)-K)^+; \theta^*\right] &= I\!\!E\left[e^{-rT}(S(T)-K)\mathbf{1}_{\{S(T)>K\}}; \theta^*\right] \\ &= e^{-rT}\left\{I\!\!E\left[S(T)\mathbf{1}_{\{S(T)>K\}}; \theta^*\right] - K I\!\!E\left[K\mathbf{1}_{\{S(T)>K\}}; \theta^*\right]\right\}. \end{aligned}$$

To evaluate the first term we apply the factorisation formula with $k = 1, h = \theta^*$ and $g(x) = \mathbf{1}_{\{x>K\}}$ and get

$$\begin{aligned} I\!\!E\left[S(T)\mathbf{1}_{\{S(T)>K\}}; \theta^*\right] &= I\!\!E\left[S(T); \theta^*\right] I\!\!E\left[\mathbf{1}_{\{S(T)>K\}}; \theta^*+1\right] \\ &= I\!\!E\left[e^{-rT}S(T); \theta^*\right] e^{rT} I\!\!P\left[S(T)>K; \theta^*+1\right] \\ &= S(0)e^{rT} I\!\!P\left[S(T)>K; \theta^*+1\right], \end{aligned}$$

where we used the martingale property of $e^{-rt}S(t)$ under the risk-neutral Esscher measure for the last step. Now the pricing formula for the European call becomes

$$S(0) I\!\!P\left[S(T)>K; \theta^*+1\right] - e^{-rT} K I\!\!P\left[S(T)>K; \theta^*\right]. \tag{27}$$

We now can use formula (27) to compute the value of a European call with strike K and maturity T. Denote the density of $\mathcal{L}(Z^{\zeta,\delta}(t))$ by $f_t^{\zeta\delta}$ (compare (5) for the exact form). Then

$$\begin{aligned} E\left[e^{-rT}(S_T-K)^+; \theta^*\right] &= S(0)\int_c^\infty f_T^{\zeta,\delta}(x; \theta^*+1)dx \\ &\quad -e^{-rT}K\int_c^\infty f_T^{\zeta,\delta}(x; \theta^*)dx, \end{aligned} \tag{28}$$

where $c = \log(K/S(0))$.

Eberlein and co-authors [16, 17] compare option prices obtained from the Black-Scholes model and prices found using the hyperbolic model with market prices. They find that the hyperbolic model provides very accurate prices and a reduction of the smile effect observed in the Black-Scholes model.

4.5 Risk-management: Hedging and Value-at-Risk

We consider hedging first, and focus on computing the standard hedge parameters, i.e. the so-called greeks. It is relatively easy to compute the delta of the European call C using formula (28). Now

$$\Delta = \frac{dC}{dS} = \int_c^\infty f_T^{\zeta,\delta}(x;\theta^*+1)dx - f_T^{\zeta,\delta}(c;\theta^*+1) + e^{-rT}\frac{K}{S} f_T^{\zeta,\delta}(c;\theta^*).$$

Consider the last two terms. Using subsequently the definition of $f_T^{\zeta,\delta}(.;.)$ and θ^* we get

$$\begin{aligned}
&-f_T^{\zeta,\delta}(c;\theta^*+1) + e^{-rT}\tfrac{K}{S} f_T^{\zeta,\delta}(c;\theta^*) \\
= &\quad -\frac{e^{c(\theta^*+1)} f_T^{\zeta,\delta}(c)}{M(\theta^*+1)^T} + e^{-rT}\frac{K}{S}\frac{e^{c\theta^*} f_T^{\zeta,\delta}(c)}{M(\theta^*)^T} \\
= &\quad -\frac{K}{S}\frac{e^{c\theta^*} f_T^{\zeta,\delta}(c)}{M(\theta^*+1)^T} + e^{-rT}\frac{K}{S}\frac{e^{c\theta^*} f_T^{\zeta,\delta}(c)}{e^{-rT}M(\theta^*+1)^T} \\
= &\quad 0.
\end{aligned}$$

So we end up with the simple expression

$$\Delta = \int_c^\infty f_T^{\zeta,\delta}(x;\theta^*+1)dx.$$

Other sensitivity parameters can be computed in a similar fashion; however, as above the evaluation has to be done numerically.

We study aspects of risk-management in terms of Value-at-Risk in a simple linear position in the underlying asset. We compare a normal fit and a full hyperbolic fit with a tail approximation via Extreme-Value theory. In particular, to compute high quantiles we use the Peak-over-Threshold method, which is outlined in

Embrechts, Klüppelberg and Mikosch [19] §6.5. (A detailed study using the POT method has been done by McNeil [36].) As may be seen from the accompanying table, the EVT quantiles obtained using Extreme-Value theory are more accurate than either the normal or hyperbolic quantiles. This is to be expected: Extreme-Value methods, being specifically designed for the tails, outperform other methods there. By contrast, the hyperbolic approach is designed to give a reasonable fit throughout, and in particular a better fit overall than the normal.

Quantile	empirical	normal	hyperbolic	EVT
0.1 %	-0.04743	-0.03670	-0.06257	-0.05388
0.5 %	-0.03490	-0.03051	-0.04927	-0.04180
1 %	-0.03004	-0.02751	-0.04138	-0.03655
5 %	-0.01873	-0.01931	-0.02543	-0.02430
95 %	0.01863	0.02028	0.02626	0.01863
99 %	0.03137	0.02848	0.045302	0.03140
99.5 %	0.03541	0.03148	0.053259	0.03894
99.9 %	0.06861	0.03767	0.069521	0.06332

Table 1: Comparison of Quantiles

5 Conclusion

The hyperbolic model has a good case to be regarded as the model of first choice in any situation where the benchmark normal, or Black-Scholes, model is found inadequate. It has a sound theoretical basis, the independent-increments assumption being the one most open to question. Also, in its four-parameter and two-parameter forms, it has a suitable set of readily interpretable parameters. Thanks to the already developed software [10], fitting the model empirically to actual data is quick and convenient. It gives a reasonable fit throughout, but is outperformed by methods based on extreme-value theory in the tails. (More examples can be found on the webside of the Freiburg Center for Data Analysis and Modelling, http://www.fdm.uni-freiburg.de/UK/).

Acknowledgement. To perform the Peaks-over-Threshold analysis in $4.5 we used the software EVIS written by A.McNeil. The software can be used in an S-Plus framework and may be downloaded from www.math.ethz.ch/mcneil/software.html.

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