pfsl13.tex
Lecture 13. 7.11.2012

Definition. Two distribution functions F, G have the same *type* if

$$G(x) = F(a + bx)$$

for some a, b. Then if $Y := (X - a)/b$, and $X \sim F$, then $Y \sim G$.

Stable laws.

The possible limit laws obtainable from centred and scaled random walks are the *stable* laws, which form the subclass S of SD. To within type (so we can take $a = 0$), they have two parameters, the *index* $\alpha \in (0, 2]$ and the *skewness parameter* $\beta \in [-1, 1]$. If $\alpha = 2$, the law is (standard) *normal*, and then $\sigma = 1$, $\nu = 0$. If $0 < \alpha < 2$, then $\sigma = 0$ and, for some $p \in [0, 1]$ and $q := 1 - p$ (the usual notation for Bernoulli trials $B(p)$), the Lévy measure has the form

$$d\nu = p\ dx/x^{1+\alpha} \quad \text{on } (0, \infty), \qquad q\ dx/|x|^{1+\alpha} \quad \text{on } (-\infty, 0),$$

while the skewness parameter ('tail-balance parameter') is

$$\beta = p - q\ (= 2p - 1)$$

(here $p + q = 1$, but this is a restriction of type, for convenience only – any value $p + q \in (0, \infty)$ will do). The CFs are

$$\phi(t) = \exp\{-\frac{1}{2}t^2\} \qquad (\text{normal case}, \alpha = 2);$$

$$\phi(t) = \exp\{-|t|^\alpha(1 - i\beta(sgn\ t)\tan\frac{1}{2}\pi\alpha)\} \quad (0 < \alpha < 1 \text{ or } 1 < \alpha < 2, -1 \leq \beta \leq 1);$$

$$\phi(t) = \exp\{-|t|(1 + i\beta(sgn\ t)\frac{2}{\pi}\log|t|\} \quad (\alpha = 1, -1 \leq \beta \leq 1).$$

If $\beta = 0$, the law is *symmetric* (X and $-X$ have the same distribution), and we obtain the *symmetric stable* laws with CFs

$$\phi(t) = \exp\{-|t|^\alpha\} \qquad (0 < \alpha \leq 2).$$

Densities.

For $\alpha = 2$, we obtain the standard normal law, whose density $e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$

we know.

For $\alpha = 1$, we obtain the *symmetric Cauchy* law, whose density

$$f(x) = \frac{1}{\pi(1+x^2)}$$

we studied above.

For $\beta = +1$, $\alpha = \frac{1}{2}$, we obtain the *Lévy density* (Problems),

For other parameter values, there is no explicit formula, but one can obtain series expansions.

One-sided stable laws.

The Lévy measure is also called the *spectral measure*. When $\beta = +1$, $p = 1, q = 0$, all its mass is on the *positive* half-line – the *spectrally positive* case: only *positive jumps*, and decrease takes place continuously, rather than by jumping. Similarly for the *spectrally negative* case $\beta = -1$, $p = 0, q = 1$: all mass on the *negative* half-line; only negative jumps.

Stable laws and tails

It is a general property of id laws F that their tail behaviour is similar to that of their Lévy measures ν. Stable laws have finite ath moments for $a < \alpha$ and infinite ath moment for $a > \alpha$. Thus for $\alpha = 2$ (normal case) all moments are finite and the CF is entire; for $1 < \alpha < 2$ the mean exists but the variance does not; for $0 < \alpha < 1$ the mean does not exist. Such behaviour is described as having *heavy tails*. These are important, in at least two areas:

1. *Insurance*. It is the *large claims* that are dangerous for an insurance company – indeed, potentially lethal. The frequency of large claims is governed by the tail decay.
2. *Finance*. The standard benchmark model of mathematical finance, the *Black-Scholes(-Merton) model* has normal (actually, log-normal) tails. But most real financial data show much fatter tail behaviour than this. Stable laws have been used to model tails of financial data. So too have Student t-distributions (which, unlike stable laws, are not restricted to $\alpha \leq 2$).

IV. NORMAL DISTRIBUTION THEORY

1. Regression

In regression (see e.g. [BF]), we have data $y_1, \ldots, y_n$, arranged as a column n-vector y. We seek to explain the data parsimoniously, in terms of p parameters $\beta_1, \ldots, \beta_p$ (arranged as a column p-vector β), via linear combinations of explanatory variables (predictor variables, covariates, regressors, ...), plus some error, which we model as an n-vector ϵ, whose components ϵ_i are assumed iid $N(0, \sigma^2)$. Then the *model equation* is

$$y = A\beta + \epsilon. \qquad (ME)$$

Here the matrix A is $n \times p$; $p << n$ ("p is much less than n") – n is the sample size (the larger the better), p the number of parameters (as small as possible, by the Principle of Parsimony); A is called the *design matrix*. We restrict attention to the case when A has *full rank*, p (otherwise, eliminate superfluous regressors to reduce to this). From the model equation

$$y_i = \sum\nolimits_{j=1}^{p} a_{ij}\beta_j, \qquad \epsilon_i \quad iid \quad N(0, \sigma^2),$$

the likelihood is

$$\begin{aligned} L &= \frac{1}{\sigma^n 2\pi^{\frac{1}{2}n}} . \prod\nolimits_{i=1}^{n} \exp\{-\frac{1}{2}(y_i - \sum\nolimits_{j=1}^{p} a_{ij}\beta_j)^2/\sigma^2\} \\ &= \frac{1}{\sigma^n 2\pi^{\frac{1}{2}n}} . \exp\{-\frac{1}{2}\sum\nolimits_{i=1}^{n}(y_i - \sum\nolimits_{j=1}^{p} a_{ij}\beta_j)^2/\sigma^2\}, \end{aligned}$$

and the log-likelihood is

$$\ell := \log L = const - n \log \sigma - \frac{1}{2}[\sum\nolimits_{i=1}^{n}(y_i - \sum\nolimits_{j=1}^{p} a_{ij}\beta_j)^2]/\sigma^2.$$

As before, we use Fisher's Method of Maximum Likelihood, and maximise with respect to β_r: $\partial\ell/\partial\beta_r = 0$ gives

$$\sum\nolimits_{i=1}^{n} a_{ir}(y_i - \sum\nolimits_{j=1}^{p} a_{ij}\beta_j) = 0 \qquad (r = 1, \ldots, p),$$

or

$$\sum\nolimits_{j=1}^{p}(\sum\nolimits_{i=1}^{n} a_{ir}a_{ij})\beta_j = \sum\nolimits_{i=1}^{n} a_{ir}y_i.$$

Write $C = (c_{ij})$ for the $p \times p$ matrix

$$C := A^T A,$$

(called the *information matrix*), which we note is *symmetric*: $C^T = C$. Then

$$c_{ij} = \sum_{k=1}^{n} (A^T)_{ik} A_{kj} = \sum_{k=1}^{n} a_{ki} a_{kj}.$$

So this says

$$\sum_{j=1}^{p} c_{rj} \beta_j = \sum_{i=1}^{n} a_{ir} y_i = \sum_{i=1}^{n} (A^T)_{ri} y_i.$$

In matrix notation, this is

$$(C\beta)_r = (A^T y)_r \qquad (r = 1, \ldots, p),$$

or combining,

$$C\beta = A^T y, \qquad C := A^T A. \tag{NE}$$

These are the *normal equations.*

As A has full rank, C is *positive definite* ($x^T C x > 0$ for all vectors $x \neq 0$) ([BF], Lemma 3.3), so we can solve the normal equations to obtain our *least-squares estimates* of β, namely

$$\hat{\beta} = C^{-1} A^T y.$$

Write

$$P := AC^{-1}A^T$$

for the *projection matrix* of A. Note that

$$P^2 = AC^{-1}A^T AC^{-1}A^T = AC^{-1}CC^{-1}A^T = AC^{-1}A^T = P,$$

so P is *idempotent*, i.e. is a *projection* (see e.g. [BF], Lemma 3.18). Also, as C is symmetric,

$$P^T = A(C^{-1})^T A^T = A(C^T)^{-1}A = AC^{-1}A^T = P :$$

P is *symmetric*, so is a *symmetric projection*. Similarly, so is $I - P$.

Call a linear transformation $P : V \to V$ a *projection onto* V_1 *along* V_2 if V is the direct sum $V = V_1 \oplus V_2$, and if $x = (x_1, x_2)^T$ with $Px = x_1$. Then (check) $I - P$ is a projection onto V_2 along V_1. Also

$$P(I - P) = P - P^2 = P - P = 0 :$$

P, $I - P$ are *orthogonal projections.*