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That is, if you know where you are (at time t), how you got there doesn't matter so far as predicting the future is concerned. Equivalently, the Markov property says that past and future are conditionally independent given the present. X is said to be *strong Markov* if the above holds with the fixed time t replaced by a stopping time τ (a random variable). This is a real restriction of the Markov property in the continuous-time case (though not in discrete time). Perhaps the simplest example of a Markov process that is not strong Markov is

$$X(t) := 0 \quad (t \leq \tau), \quad t - \tau \quad (t \geq \tau),$$

with τ exponentially distributed. Then X is Markov (from the lack of memory property of the exponential distribution), but not strong Markov (the Markov property fails at the stopping time τ). The strong Markov property to fail in cases, as here, when 'all the action is at random times'. Another example of a process Markov but not strong Markov is a left-continuous Poisson process – obtained by taking a Poisson process and making its paths left- rather than right-continuous.

Diffusions

A *diffusion* is a path-continuous strong Markov process such that for each time t and state x the following limits exist:

$$\mu(t,x) := \lim_{h \downarrow 0} \frac{1}{h} E[(X(t+h) - X(t))|X(t) = x],$$

$$\sigma^2(t,x) := \lim_{h \downarrow 0} \frac{1}{h} E[(X(t+h) - X(t))^2|X(t) = x].$$

Then $\mu(t,x)$ is called the *drift*, $\sigma^2(t,x)$ the *diffusion coefficient*.

The term diffusion derives from physical situations involving Brownian motion. The mathematics of heat diffusing through a conducting medium (which goes back to Fourier in the early 19th century) is intimately linked with Brownian motion (the mathematics of which is 20th century).

The theory of diffusions can be split according to dimension. In one dimension, there are a number of ways of treating the theory. In higher dimension, there is basically one way: via the stochastic differential equation methodology (or its reformulation in terms of a martingale problem). This shows the best way to treat the one-dimensional case: the best method is the one that generalizes. It also shows that Markov processes and martingales, as well as being the two general classes of stochastic process with which one can get anywhere mathematically, are also intimately linked technically. We will encounter diffusions largely as solutions of stochastic differential equations.

VI. LÉVY PROCESSES

1. Brownian motion

Brownian motion originates in work of the botanist Robert Brown in 1828. It was introduced into finance by Louis Bachelier in 1900, and developed in physics by Albert Einstein in 1905.

The fact that Brownian motion *exists* is quite deep, and was first proved by Norbert WIENER (1894–1964) in 1923. In honour of this, Brownian motion is also known as the *Wiener process*, and the probability measure generating it – the measure P^* on $C[0,1]$ (one can extend to $C[0,\infty)$) by

$$P^*(A) = P(W_. \in A) = P(\{t \to W_t(\omega)\} \in A)$$

for all Borel sets $A \in C[0,1]$ – is called *Wiener measure.*

Definition. A stochastic process $X = (X(t))_{t\geq 0}$ is a standard (one-dimensional) Brownian motion, BM or $BM(\mathbf{R})$, on some probability space $(\Omega, \mathcal{F}, \mathcal{P})$, if
(i) $X(0) = 0$ a.s.,
(ii) X has *independent increments*: $X(t+u)-X(t)$ is independent of $\sigma(X(s) : s \leq t)$ for $u \geq 0$,
(iii) X has *stationary increments*: the law of $X(t+u) - X(t)$ depends only on u,
(iv) X has *Gaussian increments*: $X(t+u) - X(t)$ is normally distributed with mean 0 and variance u, $X(t+u) - X(t) \sim N(0, u)$,
(v) X has *continuous paths*: $X(t)$ is a continuous function of t, i.e. $t \to X(t,\omega)$ is continuous in t for all $\omega \in \Omega$.

We can relax path continuity in (v) by assuming it only a.s.; we can then get continuity by excluding a suitable null-set from our probability space.

We denote standard Brownian motion $BM(\mathbf{R})$ by $W = (W(t))$ (W for Wiener), though $B = (B(t))$ (B for Brown) is also common. Standard Brownian motion $BM(\mathbf{R}^d)$ in d dimensions is defined by $W(t) := (W_1(t), \ldots, W_d(t))$, where W_i are independent copies of $BM(\mathbf{R})$.

We turn next to Wiener's theorem, on existence of Brownian motion.

Theorem (Wiener, 1923). Brownian motion exists.