pfsl28.tex

Lecture 28. 12.12.2012

5. Continuous time

Renewal theory.

Imagine a public room which is in permanent use, so needs to be permanently lit. At time 0, a new lightbulb is installed. It is used until it fails (at time T_1, say), and then immediately replaced by another of the same kind. This is used until it fails, after some further time-lapse T_2, say, when it is replaced, etc. The time-axis $[0, \infty)$ is thus punctuated by *renewal epochs* $T_1, T_1 + T_2, \ldots, T_1 + \ldots + T_k, \ldots$, where T_k is the lifetime of the kth lightbulb. We assume that the T_k are independent, each with distribution function F (concentrated on $(0, \infty)$, so $F(0) = 0$) and density f. So F is the *lifetime distribution*. We often focus instead on its *tail*, or *survival function*,

$$\bar{F}(x) := 1 - F(x), \qquad (x > 0).$$

So, writing X for a typical lifetime,

$$F(x) = \int_0^x f(y)dy = P(X \le x), \qquad \bar{F}(x) = \int_x^\infty f(y)dy = P(X > x).$$

Similarly for any component that must be replaced immediately on failure.

We are interested in the residual life left in the current lightbulb. Suppose it has been in use (without failure, understood) for time x. Of particular interest is the probability that it will fail within some short further time-lapse dx:

$$P(X \in (x, x + dx) | X > x).$$

This is

$$P(X \in (x, x + dx))/P(X > x) = \int_x^{x+dx} f(y)dy/\bar{F}(x) = f(x)dx/\bar{F}(x),$$

to first order in dx. It is natural to think of the coefficient of dx on the right as a *rate*. It is called the *hazard rate*, or *failure rate*, $h(x)$:

$$h(x) := f(x)/(1 - F(x)) = f(x)/\bar{F}(x) = f(x)/\int_x^\infty f(y)dy.$$

Now $-h$ is the derivative of $\log(1 - F)$, from above. So integrating,

$$1 - F(x) = \bar{F}(x) = \int_x^\infty f(y)dy = \exp\{-\int_0^x h(y)dy\} \qquad (x > 0).$$

For lightbulbs etc.: would you prefer the component you are using to be
(a) new (suggesting that it may therefore be expected to last longer), or
(b) used (suggesting that it is a good component – it has demonstrated this by surviving the test of use)?

Both views are reasonable. There is in fact a whole subject of Reliability Theory, which uses acronyms such as NBU for 'new better than used', etc.

Example. Take the simplest case possible – that of *constant hazard rate*,

$$h(x) \equiv \lambda, \qquad F(x) = 1 - e^{-\lambda x}, \qquad f(x) = \lambda e^{-\lambda x} \qquad (x > 0):$$

F is the *exponential* distribution with *parameter* λ, $F = E(\lambda)$. It turns out that this simplest possible case is also by far the most important case!

Because we now have the interpretation of the parameter λ in the exponential distribution $E(\lambda)$ as a hazard rate, we may (and will) refer to $E(\lambda)$ as the exponential distribution with *rate* λ. This is intimately linked to the Poisson process with rate, or parameter, λ, $Ppp(\lambda)$, and the Poisson distribution $P(\lambda)$.

Exponential distributions and lack of memory.

For our lightbulb, the conditional probability that it is still working at time $s + t$, given that it is working at time s, is

$$P(T > t{+}s | T > s) = P(T > t{+}s \& T > s)/P(T > s) = P(T > t{+}s)/P(T > s).$$

Suppose now that the bulbs show *no ageing* – or, to use the alternative description, have the *lack of memory* property. Then the situation above is equivalent to that of a new lightbulb surviving for (at least) time t. That is, absence of ageing is equivalent to

$$P(T > t{+}s)/P(T > s) = P(T > t): \quad P(T > t{+}s) = P(T > t).P(T > s) \qquad (s, t > 0).$$

That is, absence of ageing is equivalent to

$$\bar{F}(t + s) = \bar{F}(t).\bar{F}(s) \qquad (s, t > 0). \qquad (*)$$

This is the *Cauchy functional equation*, for which we seek a *bounded* solution. The only ones are the obvious ones – the exponential distributions, $F = E(\lambda)$ for $\lambda > 0$. For, in this case $\bar{F}(x) = e^{-\lambda x}$, and $(*)$ just says that $e^{a+b} = e^a.e^b$, the defining property of the exponential function. Summarizing this:

Theorem. The only lifetime distributions satisfying the lack of memory property $(*)$ are the exponential distributions $E(\lambda)$.

Note. This lack of memory property of the exponential distributions is the essence of the *Markov property* in continuous time. Here the *holding time* at a particular state, k say – the length of time we stay there – is exponential $E(\lambda_k)$. The main ingredients of a Markov chain in continuous time are
(i) the *jump rates* λ_k,
(ii) the *jump law*, telling us where we jump to when we jump from state k.
Renewal Theory and the Poisson Process.

If we write $X_1, X_2, \ldots$ for the lifetimes of the first, second, ... lightbulbs (denoted $T_1, T_2, \ldots$ above), write

$$S_n := X_1 + \ldots + X_n$$

($S_0 := 0$) for the nth partial sum. Then $S = (S_n)_{n=0}^{\infty}$ is a random walk, with step-length distribution the lifetime distribution F. Now write

$$N_t, \quad \text{or} \quad N(t), := \max\{k : S_k \le t\} \qquad (t \ge 0).$$

Then N_t is the number of failures, or replacements, or *renewals*, up to time t. Note that N_t is a random function of time t, which starts at 0 and jumps upward by 1 at the epochs $S_1, S_2, \ldots$ of successive renewals. The process

$$N := \{N_t : t \ge 0\}$$

is called a *renewal process*, with *lifetime distribution* F.

We have already seen that the exponential distributions $E(\lambda)$ play a specially important role among lifetime distributions. The corresponding renewal process is λ, $Ppp(\lambda)$.

Theorem. Among renewal processes, the Poisson processes are the only ones that have the Markov property.

Proof. As in discrete time, the Markov property means that, to predict the future, all that matters is the present, not the past. The present is just the count of which lightbulb is in current use. *How long* the current lightbulb has been in use refers to the past. The Markov property holds iff this is irrelevant, i.e. iff the lifetimes have the lack-of-memory property (show no ageing), i.e. iff F is an exponential distribution $E(\lambda)$ – the Poisson case. //

Explosions; compactification

Recall from VI that a Lévy process (with infinite mass in its Lévy measure) makes infinitely many jumps in finite time. In the same way, a Markov chain in continuous time can make infinitely many jumps in finite time – a phenomenon known as *explosion.* This leads to many complications. Even though there may be infinitely many states, explosion can 'take the process through all of them', to some new "infinite state". The question then arises of whether to stop or kill the process at this point – in which case we have a process for which time stops – or if not, how to "re-start the process". We then choose the new starting point from some *entrance law.*

You will have encountered the symbols $+\infty$, $-\infty$ used on the real line $\mathbf{R}$, and their use to give the *extended real line* $\bar{\mathbf{R}}$, and perhaps also the *extended complex plane* $\bar{\mathbf{C}}$, e.g. via *stereographic projection* (see e.g. M2P3 L4). Here $\mathbf{C}$, which is not compact, is embedded in the extended complex plane, which is compact, by adding a *compactification point*, or "point at infinity".

One needs to do similar things with Markov chains in continuous time, if they have explosions, or various other types of pathological behaviour. We *compactify* the state-space suitably, by adding "points at infinity"; the compactified chain (or rather, process) may be free of pathological behaviour. One such process is *Ray-Knight compactification.* The "extra" points are called the *boundary*; one such is the *Martin boundary.* One classic pathological chain is the *Feller-McKean chain.* All this leads us well beyond MSc level, so we must leave it there. But it does explain why the treatment of continuous-time Markov chains in books at this level (e.g., [GS] Ch. 6) is sketchy compared to that of discrete-time Markov chains.

Continuous time and continuous state space

Markov processes are named after A. A. MARKOV (1856-1922) in 1907. He worked with discrete time and a finite state space. Countably infinite state spaces were studied by Kolmogorov in 1936, and W. DÖBLIN (Doeblin) (1915-1940) in 1937. The term *Markov chain* used to denote discrete (finite or countable) state space, but more modern usage, followed in [MT], uses Markov chain to denote discrete time and Markov process for continuous time. The extra difficulty of the continuous-time case is mentioned above. The change in usage reflects the progress made in extending the discrete-state theory to the continuous-state context, for which see [MT]. This progress has is highly relevant to (and was partly motivated by) MCMC.