PROBABILITY FOR STATISTICS: MOCK EXAM, 2012

Q1. Define the *Chi-square distribution* with n *degrees of freedom*, $\chi^2(n)$. Show that it has
(i) mean n, variance $2n$;
(ii) characteristic function $1/(1-2it)^{\frac{1}{2}n}$;
(iii) density

$$f(x) = \frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)} x^{\frac{1}{2}n-1} \exp(-\frac{1}{2}x) \qquad (x > 0).$$

Q2. Define the *multivariate normal distribution.* Find the characteristic function of the multivariate normal distribution $N(\mu, \Sigma)$ in n dimensions with mean vector μ and covariance matrix Σ.

Find the condition for independence of the linear forms Ax, Bx, with A, B $n \times n$ matrices and $x \sim N(\mu, \Sigma)$.

Q3. The *Lévy density* is defined by

$$f(x) := \frac{1}{\sqrt{2\pi}} . \frac{1}{x^{3/2}} . \exp(-\frac{1}{2x}) \qquad (x > 0).$$

Show that f is a density, and that its Laplace-Stieltjes transform is

$$\psi(s) = e^{-\sqrt{2s}} \qquad (s \geq 0).$$

Q4. Show that

$$\pi_i = \binom{d}{i}^2 \Big/ \binom{2d}{d}$$

defines a probability distribution on $\{0, 1, \ldots d\}$ (the *hypergeometric distribution*, $HG(d)$).

In the Bernoulli-Laplace urn model, there are $2d$ balls, d black and d white, and two urns, each containing d balls. At each stage, a ball is chosen from each urn and they are interchanged; the state is the number i of white balls in the first urn. Show that the transition matrix $P = (p_{ij})$ is given by

$$p_{i,i-1} = (i/d)^2, \quad p_{i,i} = 2d(d-i)/d^2, \quad p_{i,i+1} = (d-i)^2/d^2, \quad p_{i,j} = 0 \quad \text{otherwise}.$$

Show that the chain is reversible, and has invariant distribution $HG(d)$.

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