pfsprob10.tex

PROBLEMS 10 13.12.2012

Q1 (*Bernoulli-Laplace urn model*). By adapting the method of Problems 9 Q3, show that the Bernoulli-Laplace urn is
(i) reversible/has detailed balance;
(ii) has invariant distribution the *hypergeometric distribution*

$$\pi_i = \binom{d}{i}^2 / \binom{2d}{d}$$

(this is a probability distribution, by Problems 7 Q3).

Q2 (*Bernoulli-Laplace urn* continued). Show that the mean recurrence time of state 0 is

$$\mu_0 = \binom{2d}{d} \sim 4^d/\sqrt{\pi d} \qquad (d \to \infty).$$

Interpret this in the context of Statistical Mechanics, where d is of the order of Avogadro's number, 6.02×10^{23}.

Q3 *Branching processes.* In a population model, one starts with a single ancestor (the 0th generation). On death, he is replaced by a random number Z of offspring, with PGF $P(s)$ (the 1st generation), and mean $\mu = E[Z]$. They reproduce independently and in the same way, their offspring forming the second generation, with PGF P_2, and so on. Show that:
(i) $P_2(s) = P(P(s))$, the second (functional) iterate of P;
(ii) the nth generation, of size Z_n say, has PGF P_n, the nth functional iterate of P (defined inductively by $P_n = P_{n-1}(P) = P(P_{n-1})$);
(iii) the mean generation size is $E[Z_n] = \mu^n$.

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