pfssoln2.tex

SOLUTIONS 2. 25.10.2012

Q1. (i) The joint density of the x_i is

$$f(x) = (2\pi)^{-\frac{1}{2}n} \prod_{i=1}^{n} \exp\{-\frac{1}{2}x_i^2\} = (2\pi)^{-\frac{1}{2}n} \exp\{-\frac{1}{2}\sum_{1}^{n} x_i^2\} = (2\pi)^{-\frac{1}{2}n} \exp\{-\frac{1}{2}\|x\|^2\}.$$

The Jacobian of the change of variable is the determinant |O|, = 1 as O is orthogonal (= length-preserving). So the joint density of the y_i is

$$g(y) = (2\pi)^{-\frac{1}{2}n} \exp\{-\|y\|^2\} = (2\pi)^{-\frac{1}{2}n} \exp\{-\sum_{1}^{n} y_i^2\},\$$

which says that the y_i are iid N(0, 1).

(ii) The condition for a matrix O to be orthogonal is that the rows are of length 1 and orthogonal vectors. Take the first row as e_1 , and use Gram-Schmidt orthogonalisation to find e_2 orthogonal to e_1 , then e_3 orthogonal to e_1, e_2 etc. The e_i form the rows of an orthogonal matrix with first row e_1 . (iii) Put $Z_i := (X_i - \mu)/\sigma$, $Z := (Z_1, \ldots, Z_n)^T$; then the Z_i are iid N(0, 1),

$$\bar{Z} = (\bar{X} - \mu)/\sigma, \qquad nS^2/\sigma^2 = \sum_{i=1}^{n} (Z_i - \bar{Z})^2.$$

Also

$$\sum_{1}^{n} Z_{i}^{2} = \sum_{1}^{n} (Z_{i} - \bar{Z})^{2} + n\bar{Z}^{2},$$

since $\sum_{i=1}^{n} Z_i = n \overline{Z}$. The terms on the right above are quadratic forms, with matrices A, B say, so we can write

$$\sum_{1}^{n} Z_i^2 = Z^T A Z + Z^T B X. \tag{(*)}$$

Put W := PZ with P a Helmert transformation with first row $(1, \ldots, 1)/\sqrt{n}$:

$$W_1 = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i = \sqrt{n} \bar{Z}; \qquad W_1^2 = n \bar{Z}^2 = Z^T B Z.$$

So by above,

$$Z^{T}AZ = \sum_{i=1}^{n} (Z_{i} - \bar{Z})^{2} = nS^{2}/\sigma^{2}, = \sum_{i=1}^{n} W_{i}^{2},$$

as $\sum_{1}^{n} Z_{i}^{2} = \sum_{1}^{n} W_{i}^{2}$. But the W_{i} are independent (by the orthogonality of P), so W_{1} is independent of W_{2}, \ldots, W_{n} . So W_{1}^{2} is independent of $\sum_{2}^{n} W_{i}^{2}$. So nS^{2}/σ^{2} is independent of $n(\bar{X}-\mu)^{2}/\sigma^{2}$, so S^{2} is independent of \bar{X} , as claimed.

Q2. (i) For n = 1, the mean is 1, because a $\chi^2(1)$ is the square of a standard normal, and a standard normal has mean 0 and variance 1. The variance is 2, because the fourth moment of a standard normal X is 3, and

$$var(X^2) = E[(X^2)^2] - [E(X^2)]^2 = 3 - 1 = 2.$$

For general n, the mean is n because means add, and the variance is 2n because variances add over independent summands.

(ii) For X standard normal, the MGF of its square X^2 is

$$M(t) := \int e^{tx^2} \phi(x) dx = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{tx^2} \cdot e^{-\frac{1}{2}x^2} dx$$

We see that the integral converges only for $t < \frac{1}{2}$, when it is $1/\sqrt{(1-2t)}$:

$$M(t) = 1/\sqrt{1-2t}$$
 $(t < \frac{1}{2})$ for $X \ N(0,1)$.

So by definition of $\chi^2(n)$ the MGF of a $\chi^2(n)$ is

$$M(t) = 1/(1-2t)^{\frac{1}{2}n}$$
 $(t < \frac{1}{2})$ for $X \chi^2(n)$.

(iii) First, f(.) is a density, as it is non-negative, and integrates to 1:

$$\int f(x)dx = \frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)} \cdot \int_0^\infty x^{\frac{1}{2}n-1} \exp(-\frac{1}{2}x)dx$$
$$= \frac{1}{\Gamma(\frac{1}{2}n)} \cdot \int_0^\infty u^{\frac{1}{2}n-1} \exp(-u)du \qquad (u := \frac{1}{2}x)$$
$$= 1,$$

by definition of the Gamma function. Its MGF is

$$M(t) = \frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)} \cdot \int_0^\infty e^{tx} \cdot x^{\frac{1}{2}n-1} \exp(-\frac{1}{2}x) dx = \frac{1}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)} \cdot \int_0^\infty x^{\frac{1}{2}n-1} \exp(-\frac{1}{2}x(1-2t)) dx.$$

Substitute $u := \frac{1}{2}x(1-2t)$ in the integral. One obtains

$$M(t) = (1 - 2t)^{-\frac{1}{2}n} \cdot \frac{1}{\Gamma(\frac{1}{2}n)} \cdot \int_0^\infty u^{\frac{1}{2}n - 1} e^{-u} du = (1 - 2t)^{-\frac{1}{2}n},$$

by definition of the Gamma function. //

NHB