pfssoln3.tex

SOLUTIONS 3. 1.11.2012

Q1.

$$\Gamma(x+a) \sim \sqrt{2\pi} e^{-x-a}(x+a)^{x+a-\frac{1}{2}} \sim \sqrt{2\pi} e^{-x} x^{x+a-\frac{1}{2}} \left(1+\frac{a}{x}\right)^{x+a-\frac{1}{2}}$$

$$\sim \sqrt{2\pi} e^{-x} x^{x-\frac{1}{2}} . x^a . e^{-a} \left(1+\frac{a}{x}\right)^x \sim \Gamma(x).x^a,$$

using Stirling's formula and $(1+a/x)^x \to e^a$.

Q2. (i) We show that for $t(r)$, the density $f(x) \to \phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$ as $r \to \infty$. As $(1+x/n)^n \to e^x$ as $n \to \infty$ ('compound interest'), the bracket tends to $e^{-\frac{1}{2}x^2}$.

By Q1,
$$\Gamma(r+a) \sim r^a \Gamma(r) \qquad (r \to \infty).$$

So the ratio of Γs $\sim \frac{1}{2} r^{\frac{1}{2}}$. So the constant $\sim 1/\sqrt{2\pi}$. Combining, $f(x) \to \phi(x)$, as required.

(ii) $\bar{X} \sim N(\mu, \sigma^2/n)$, so $\sqrt{n}(\bar{X}-\mu)/\sigma \sim N(0,1)$. But

$$S^2 \to \sigma^2, \qquad S \to \sigma \qquad (n \to \infty),$$

by the Law of Large Numbers ('Law of Averages' – L10), and $\sqrt{n-1} \sim \sqrt{n}$. Combining,
$$t(n-1) \to N(0,1).$$

Q3.

$$
\begin{aligned}
Q(\lambda) &= \lambda \frac{1}{n} \sum_1^n (x_i - \bar{x})^2 + 2\lambda \frac{1}{n} \sum_1^n (x_i - \bar{x})(y_i - \bar{y}) + \frac{1}{n} \sum_1^n (y_i - \bar{y})^2 \\
&= \lambda^2 \overline{(x-\bar{x})^2} + 2\lambda \overline{(x-\bar{x})(y-\bar{y})} + \overline{(y-\bar{y})^2} \\
&= \lambda^2 S_{xx} + 2\lambda S_{xy} + S_{yy}.
\end{aligned}
$$

Now $Q(\lambda) \ge 0$ for all λ, so $Q(\cdot)$ is a quadratic which does not change sign. So its discriminant is ≤ 0 (if it were > 0, there would be distinct real roots and a sign change). So ("$b^2 - 4ac \le 0$")

$$s_{xy}^2 \le s_{xx} s_{yy} = s_x^2 s_y^2, \qquad r^2 := (s_{xy}/s_x s_y)^2 \le 1.$$

So
$$-1 \leq r \leq +1.$$

The extremal cases $r = \pm 1$ or $r^2 = 1$, have discriminant 0, that is $Q(\lambda)$ has a repeated real root, λ_0 say. But then $Q(\lambda_0)$ is the sum of squares of $\lambda_0(x_i - \bar{x}) + (y_i - \bar{y})$, which is zero. So each term is 0:
$$\lambda_0(x_i - \bar{x}) + (y_i - \bar{y}) = 0 \quad (i = 1 \ldots n).$$

That is, all the points (x_i, y_i) $(i = 1 \ldots n)$, lie on a straight line through the centroid $(\bar{x}, \bar{y})$ with slope $-\lambda_0$.

Q4. Similarly
$$\begin{aligned}
Q(\lambda) &= E[\lambda^2(x - Ex)^2 + 2\lambda(x - Ex)(y - Ey) + (y - Ey)^2] \\
&= \lambda^2 E[(x - Ex)^2] + 2\lambda E[(x - Ex)(y - Ey)] + E[(y - Ey)^2] \\
&= \lambda^2 \sigma_x^2 + 2\lambda \sigma_{xy} + \sigma_y^2.
\end{aligned}$$

As before $Q(\lambda) \geq 0$ for all λ, as the discriminant is ≤ 0, i.e.
$$\sigma_{xy}^2 \leq \sigma_x^2 \sigma_y^2, \quad \rho := (\sigma_{xy}/\sigma_x \sigma_y)^2 \leq 1, \quad -1 \leq \rho \leq +1.$$

The extreme cases $\rho = \pm 1$ occur iff $Q(\lambda)$ has a repeated real root λ_0. Then
$$Q(\lambda_0) = E[(\lambda_0(x - Ex) + (y - Ey))^2] = 0.$$

So the random variable $\lambda_0(x - Ex) + (y - Ey)$ is zero (a.s. – except possibly on some set of probability 0). So all values of (x, y) lie on a straight line through the centroid (Ex, Ey) of slope $-\lambda_0$, a.s.

Note. A slight extension of this argument, using an inner product on a complex vector space, works with complex numbers and leads to conclusions of the form $|r| \leq 1$, $|\rho| \leq 1$. For details, see e.g. Section 2.3 of
D. J. H. GARLING, *Inequalities: A journey into linear analysis*, CUP, 2007. A real inner product is *bilinear*. A complex inner product is *sesquilinear*: *linear* in the first argument,
$$\langle af + bg, h \rangle = a\langle f, h \rangle + b\langle g, h \rangle,$$
but *antilinear* in the second argument:
$$\langle f, ag + bh \rangle = \bar{a}\langle f, g \rangle + \bar{b}\langle f, h \rangle.$$

In this course, we will deal mainly with *real* inner products and Hilbert space, but the *complex* case is very important. NHB